Making ice sheet models scale properly

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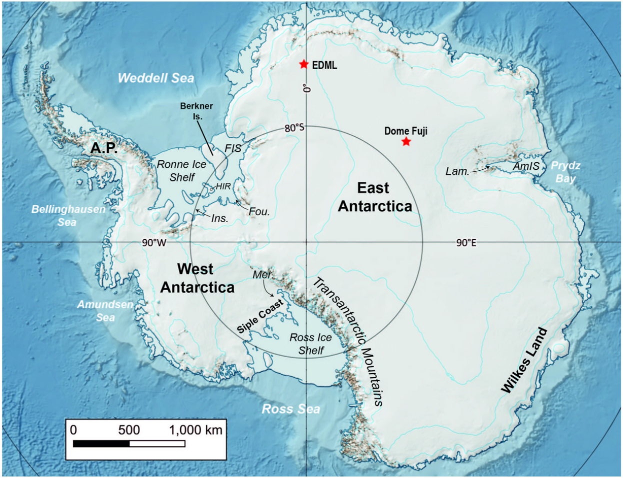
UNIVERSITY of ALASKA
Many Traditions One Alaska

Outline

- *def.* an **ice sheet** is a large glacier with small thickness/width



Antarctic ice sheet

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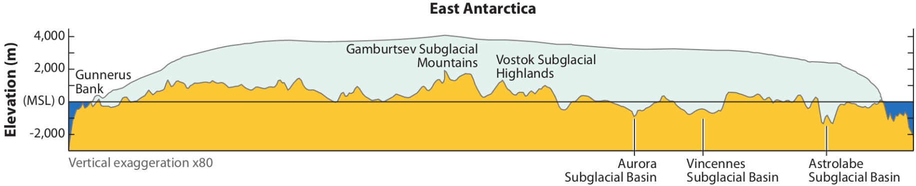

(Schoof & Hewitt 2013)

note vertical exaggeration, smooth surface, and rough bed

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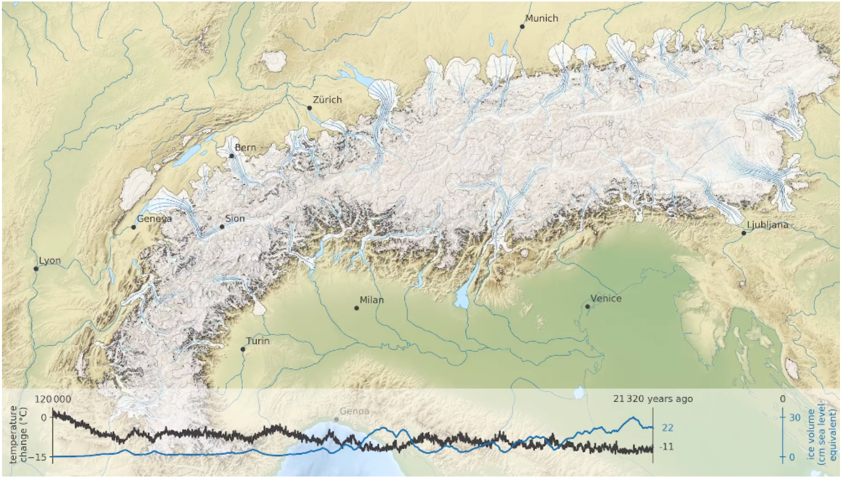

modeled Alpine ice sheet near last glacial maximum (Seguinot et al 2018)

what is an ice sheet?

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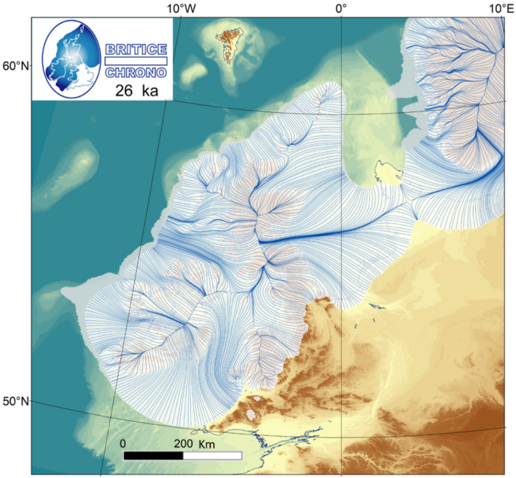

modeled British-Irish ice sheet near last glacial maximum (Clark et al 2022)

what is an ice sheet?

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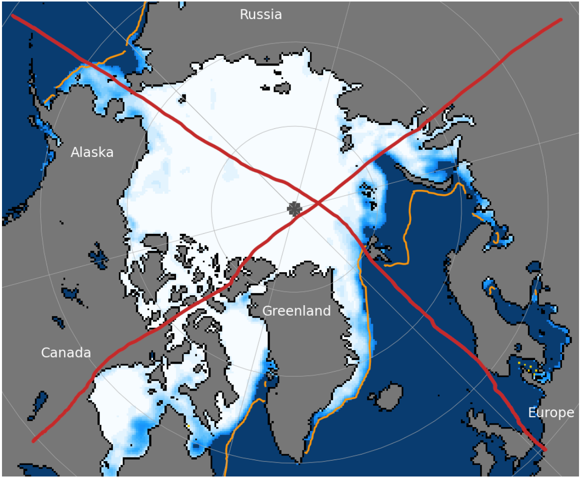
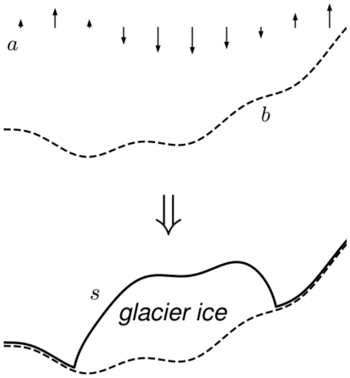


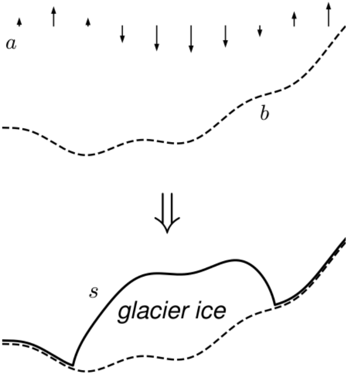
an ice sheet is *not* sea ice!

- glacier ice is modeled as a *very viscous, incompressible, non-Newtonian fluid*
  - more on that soon
- glaciers lie on *topography*
  - sometimes they float (*ice shelf*)
- glacier geometry and velocity evolve *in contact with climate*:
  - snowfall
  - surface melt
  - subglacial melt
  - sub-shelf melt (when floating)
  - calving (into ocean)



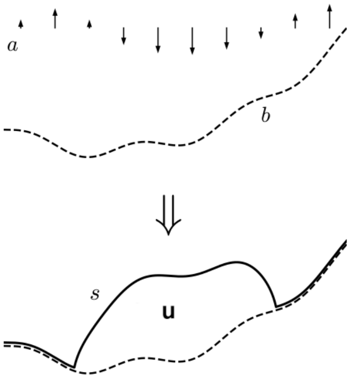
simplifications

- for simplicity/clarity of the upcoming modeling, I will ignore much of glacier physics
- **ignoring**:
  - floating ice
  - subglacial hydrology
  - ice temperature
  - fracture processes (calving, crevasses)
  - solid earth deformation
- see UAF's Parallel Ice Sheet Model (`pism.io`), for example, as a model which includes these processes



Definition

an **ice sheet model** is a map which simulates an ice sheet in a climate



- at least two inputs:
  - *surface mass balance*

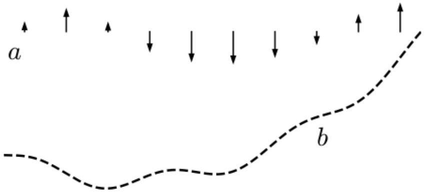
$$a(t,x,y) = \begin{pmatrix}\text{precipitation minus} \\ \text{melt \& runoff}\end{pmatrix}$$

    - units of mass flux: $\text{kg}\,\text{m}^{-2}\text{s}^{-1}$
  - *bed elevation* $b(x,y)$
- at least two outputs:
  - *upper surface elevation* $s(t,x,y)$
  - *ice velocity* $\mathbf{u}(t,x,y,z)$
- **map**: $\begin{pmatrix}\text{climate \&} \\ \text{topography}\end{pmatrix} \to \begin{pmatrix}\text{geometry} \\ \text{\& velocity}\end{pmatrix}$

- data $a(t,x,y)$, $b(x,y)$ are defined on a **fixed domain**:

$$t \in [0,T] \quad \text{and} \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

- solution surface elevation $s(t,x,y)$ is defined on $[0,T] \times \Omega$
    - also a fixed domain,
    - but $s = b$ where there is no ice
- $s(t,x,y)$ determines the time-dependent **icy domain** $\Lambda(t) \subset \mathbb{R}^3$, on which the solution velocity $\mathbf{u}(t,x,y,z)$ is defined:



$$\Lambda(t) = \{(x,y,z) \,:\, b(x,y) < z < s(t,x,y)\}$$

basic ice sheet model: notation

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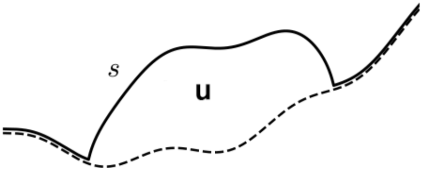

basic ice sheet model: conservation

- ice sheet evolution should conserve physical quantities:
  - mass
  - momentum
  - ~~energy~~ $\leftarrow$ *ignored for simplicity in this talk*
- conservation of mass is important both
  - in the icy domain $\Lambda(t) \subset \mathbb{R}^3$:

$$\textbf{incompressibility} \qquad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Lambda(t),$$

  - and on the ice surfaces:

$$\begin{aligned} &\textbf{surface kinematic equation (SKE)} && \frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s = a && \text{on } \Gamma_s(t), \\ &\textbf{non-penetration} && \mathbf{u}|_b \cdot \mathbf{n}_b = 0 && \text{on } \Gamma_b(t). \end{aligned}$$

  - ▷ $\Gamma_s(t), \Gamma_b(t) \subset \partial\Lambda(t)$ denote the surface and base of the ice
  - ▷ $\mathbf{n}_s = \langle -\nabla s, 1 \rangle$ is upward surface normal

- ice sheet evolution is a **free-boundary** problem for conserved quantities
- specifically, the surface kinematic equation (SKE)

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s = a$$

applies *only* on the ice upper surface $\Gamma_s(t)$

- in the remainder of the (fixed) domain $\Omega \subset \mathbb{R}^2$, **complementarity** holds:

$$s = b \quad \text{and} \quad a \le 0$$

- for more on this perspective see Bueler (2021)

- **nonlinear complementarity problem** (NCP) :

$$
\begin{aligned}
s - b &\geq 0 && \text{on } \Omega \subset \mathbb{R}^2 \\
\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &\geq 0 && \text{''} \\
(s - b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) &= 0 && \text{''} \\
-\nabla \cdot (2\nu(D\mathbf{u})\, D\mathbf{u}) + \nabla p - \rho_{\text{i}}\mathbf{g} &= \mathbf{0} && \text{in } \Lambda(t) \subset \mathbb{R}^3 \\
\nabla \cdot \mathbf{u} &= 0 && \text{''} \\
\tau_b - \mathbf{f}(\mathbf{u}|_b) &= \mathbf{0} && \text{on } \Gamma_b(t) \\
\mathbf{u}|_b \cdot \mathbf{n}_b &= 0 && \text{''} \\
(2\nu(D\mathbf{u})D\mathbf{u} - pI)\,\mathbf{n}_s &= \mathbf{0} && \text{on } \Gamma_s(t)
\end{aligned}
$$

  - $\mathbf{u}|_s = 0$ where no ice
  - viscosity by Glen law: $2\nu(D\mathbf{u}) = \Gamma|D\mathbf{u}|^{\mathrm{p}-2}$

basic ice sheet model: strong form = NCP coupled to Stokes

- nonlinear complementarity problem (NCP) coupled to **Stokes**:

$$
\begin{aligned}
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- for this slide, forget complementarity and boundary conditions
- result: SKE coupled to Stokes

$$\begin{aligned} \frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a &= 0 \\ -\nabla \cdot \left(2\nu(D\mathbf{u})\, D\mathbf{u}\right) + \nabla p - \rho_{\text{i}} \mathbf{g} &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

- only the first of these 5 equations has a time derivative
    - because ice is very viscous and incompressible
- this time-dependent problem is a **differential algebraic equation** (DAE), an extremely stiff system:

$$\begin{aligned} \dot x &= f(x,y) \\ 0 &= g(x,y) \end{aligned}$$

    - but in ∞ dimensions (PDAE?), and subject to complementarity

basic ice sheet model: is a DAE system

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basic ice sheet model: current research and thinking

- to the best of my knowledge, *no* current research groups are studying well-posedness or regularity for this basic model
  - when pressed, most researchers would agree NCP-coupled-to-Stokes is the *intended* model
  - well-posedness of the lubrication approximation of the model has been considered; existence proved in (Jouvet & Bueler 2012)
- numerical modelers tend to think of the Stokes problem separately from surface evolution
  - *time-splitting* or *explicit time-stepping* is often taken for granted
- ice sheet geometry evolution is addressed with minimal awareness of complementarity
- NCP-coupled-to-Stokes is *not yet* in common use for high-resolution, long-duration ice sheet simulations
  - because it is too slow
  - **can we make it fast enough to use?**

Outline

- thickness transport form helps for evolution or stability questions
- define:

$H(t,x,y) = s - b$	ice thickness
$\mathbf{U}(t,x,y) = \frac{1}{H}\int_b^s \mathbf{u}\,dz$	vertically-averaged horizontal velocity

  - note s and H are equivalent variables for modeling ice geometry
- the **mass continuity equation** for thickness follows from SKE and incompressibility:

$$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

- *question:* is this really an advection equation?
  answer: not really . . . ice flows (mostly) downhill so

$$\mathbf{U} \sim -\nabla s \sim -\nabla H$$

- NCP-coupled-to-Stokes DAE system has no characteristic curves

ice sheet models: the mass-continuity equation view

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advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- the diffusion schema is literal in the lubrication approximation
  - more on this momentarily
- but the fact that ice flows downhill has *time-stepping stability* consequences
  - regardless of your preference for the advective schema!
- note both forms are highly-nonlinear: $\mathbf{U}(H, \nabla s)$, $D(H, \nabla s)$

- the simplest of several shallow approximations is the "lubrication" approximation, the **shallow ice approximation** (SIA)
- SIA version of the NCP:

$$s - b \geq 0, \quad \frac{\partial s}{\partial t} + \Phi(s) - a \geq 0, \quad (s - b)\left(\frac{\partial s}{\partial t} + \Phi(s) - a\right) = 0$$

the surface motion contribution $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ has a formula:

$$\Phi(s) = -\frac{\gamma}{\mathrm{p}}(s - b)^{\mathrm{p}}|\nabla s|^{\mathrm{p}} - \nabla \cdot \left(\frac{\gamma}{\mathrm{p}+1}(s - b)^{\mathrm{p}+1}|\nabla s|^{\mathrm{p}-2}\nabla s\right)$$

  - constants $\mathrm{p} = \mathrm{n} + 1$ and $\gamma > 0$ relate to ice deformation

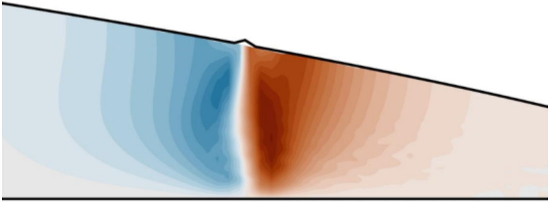
- $\Phi(s)$ resolves to a **doubly-nonlinear differential operator**
  - porous medium and p-Laplacian type simultaneously
  - *local* in surface and bed topography
  - existence is known for this NCP problem (Jouvet & Bueler, 2012), when written as a **variational inequality** weak form

nonlocality

- from now on, let us avoid shallowness approximations
- then the basic ice sheet model (NCP coupled to Stokes) problem has a **non-local** surface velocity function $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$

$$s - b \geq 0, \quad \frac{\partial s}{\partial t} + \Phi(s) - a \geq 0, \quad (s - b)\left(\frac{\partial s}{\partial t} + \Phi(s) - a\right) = 0$$

- *figure:* the Stokes velocity solution responds to a surface perturbation by up- and down-stream changes, for several ice thicknesses, while the SIA velocity responds only underneath the surface perturbation



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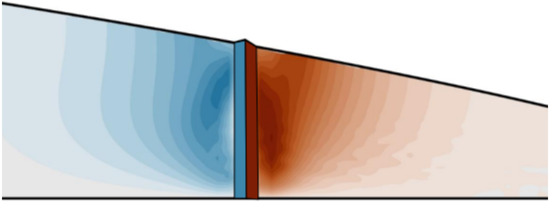


traditional PDE time-stepping

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- let us recall some traditional numerical analysis

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- **explicit** time stepping is common for **advections**
- for example, forward Euler using spacing h and time step Δt:

$$\frac{H_j^{\ell+1} - H_j^\ell}{\Delta t} + \frac{\mathbf{q}_{j+1/2}^\ell - \mathbf{q}_{j-1/2}^\ell}{h} = a_j^\ell$$

  - need good approximations of flux $\mathbf{q} = \mathbf{U} H$: upwinding, Lax-Wendroff, streamline diffusion, flux-limiters, ...
  - conditionally stable, with CFL maximum time step

$$\Delta t \le \frac{h}{\max |\mathbf{U}|} = O(h)$$

traditional PDE time-stepping

advective schema:	$\dfrac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$
diffusion schema:	$\dfrac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$

- **explicit** time stepping for **diffusions** is best avoided
- for example, forward Euler:

$$\frac{H_j^{\ell+1} - H_j^{\ell}}{\Delta t} - \frac{D_{j+\frac{1}{2}}(s_{j+1}^{\ell} + s_j^{\ell}) - D_{j-\frac{1}{2}}(s_j^{\ell} + s_{j-1}^{\ell})}{h^2} = a_j^{\ell}$$

  - conditionally stable, with maximum time step

$$\Delta t \leq \frac{h^2}{\max D} = O(h^2)$$

advective schema: $$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$$

diffusion schema: $$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$$

- **implicit** time stepping for **diffusions** is often recommended
- for example, backward Euler:

$$\frac{H_j^{\ell+1} - H_j^{\ell}}{\Delta t} - \frac{D_{j+\frac{1}{2}}(s_{j+1}^{\ell+1} + s_j^{\ell+1}) - D_{j-\frac{1}{2}}(s_j^{\ell+1} + s_{j-1}^{\ell+1})}{h^2} = a_j^{\ell}$$

  - unconditionally stable, but must solve equations at each step
  - further implicit schemes: Crank-Nicolson, BDF, ...

time-stepping in current and future ice sheet models

- current-technology large-scale models use explicit time stepping
  - this is embarrassing: the mathematical problem is a DAE
  - the accuracy/performance/usability consequences of the suppressed free-boundary/DAE/diffusive character are hard to sweep under the rug
- most researchers believe the advection schema
  - time step is determined by CFL using coupled solution velocity **U**
- **implicit time-stepping** is appropriate for DAE problems
- a sequence of NCP-coupled-to-Stokes free-boundary problems must be solved at each time step

Outline

the Stokes problem for ice

- a non-shallow model solves a Stokes problem at each step:

$$\begin{aligned}
-\nabla\cdot(2\nu(D\mathbf{u})\,D\mathbf{u})+\nabla p-\rho_{\text{i}}\mathbf{g}&=\mathbf{0} && \text{in } \Lambda\subset\mathbb{R}^3\\
\nabla\cdot\mathbf{u}&=0 && \text{''}\\
\boldsymbol{\tau}_b-\mathbf{f}(\mathbf{u}|_b)&=\mathbf{0} && \text{on } \Gamma_b\\
\mathbf{u}|_b\cdot\mathbf{n}_b&=0 && \text{''}\\
(2\nu(D\mathbf{u})D\mathbf{u}-pI)\,\mathbf{n}_s&=\mathbf{0} && \text{on } \Gamma_s
\end{aligned}$$

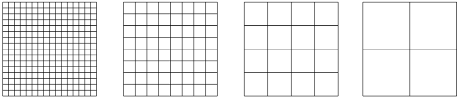
- this is the **stress balance** (conservation of momentum) problem which determines velocity **u** and pressure p
- how fast is the numerical solution process?
  - how do solution algorithms **scale** with increasing spatial resolution?

summary: PDE solver algorithmic scaling

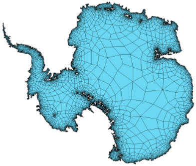
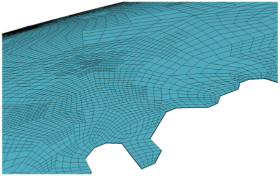
- for example, consider the 2D Poisson equation:

$$-\nabla^2 u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

- discretization generates a linear system $A\mathbf{u} = \mathbf{b}$ with $\mathbf{u} \in \mathbb{R}^m$
- data size m is the number of unknowns
  - for low-order discretizations, m = #(nodes in the grid)
  - m scales with mesh cell diameter h: $m \sim h^{-2}$ in 2D
- **complexity** or **algorithmic scaling** of flops, as $m \to \infty$, depends on solver algorithm:
  - $O(m^3)$ for direct linear algebra, ignoring matrix structure
  - $\approx O(m^2)$ for sparsity-exploiting direct linear algebra
  - $O(m^1)$, **optimal**, e.g. for **multigrid** solvers (below)



- Stokes: m = #(velocity and pressure unknowns)
- model the scaling as $O(m^{1+\alpha})$, with $\alpha = 0$ optimal
- **near-optimal solvers** already exist: $\leftarrow$ *good news!*
  - $\alpha = 0.08$ for Isaac et al. (2015) Stokes solver
    - unstructured quadrilateral/tetrahedral mesh, $Q_k \times Q_{k-2}$ stable elements, Schur-preconditioned Newton-Krylov, ice-column-oriented algebraic multigrid (AMG) preconditioner for $(\mathbf{u}, \mathbf{u})$ block

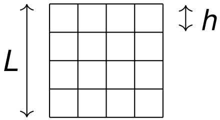
  - $\alpha = 0.05$ for Tuminaro et al (2016) 1st-order (shallow) AMG solver
  - similar for Brown et al (2013) 1st-order (shallow) GMG solver

Outline

- ice sheets are thin layers, thus ice sheet models often have $O(1)$ mesh points in the vertical direction
  - e.g. Issac et al (2015) Stokes solver
  - *simple message*: I am ignoring refinement in the vertical

- let m = #(surface elevation & velocity & pressure unknowns)
- for map-plane domain $\Omega \subset \mathbb{R}^2$ of width L and cells of diameter h:

$$m \sim \frac{L^2}{h^2}$$



- explicit time-stepping schemata:

advective	$\frac{\partial H}{\partial t} + \nabla \cdot (\mathbf{U} H) = a$	$\Delta t \leq \frac{h}{U}$
diffusion	$\frac{\partial H}{\partial t} - \nabla \cdot (D \nabla s) = a$	$\Delta t \leq \frac{h^2}{D}$

- stress-balance solver scaling parameterized as $O(m^{1+\alpha})$

- glaciologists want to run time-stepping high-resolution simulations of ice sheets over e.g. 10^5 year ice age cycles
- proposed metric: **flops per model year**
- the question:

  how does this metric **scale** in the **high spatial resolution limit** $h \to 0$, equivalently $m \to \infty$?

- the goal: $O(h^{-2}) = O(m^1)$

explicit ice sheet model performance

time-stepping		*flops per model year*
explicit	SIA	$O\left(\frac{D\,L^2}{h^4}\right) = O\left(\frac{D}{L^2} m^2\right)$
explicit (*advective*)	Stokes	$O\left(\frac{U\,L^{2+2\alpha}}{h^{3+2\alpha}}\right) = O\left(\frac{U}{L} m^{1.5+\alpha}\right)$
(*diffusive*)	Stokes	$O\left(\frac{D\,L^{2+2\alpha}}{h^{4+2\alpha}}\right) = O\left(\frac{D}{L^2} m^{2+\alpha}\right)$

- explicit time-stepping implies **many stress-balance solves**
  - while stress-balance scaling exponent α is important, even optimality ($\alpha = 0$) cannot rescue performance

implicit time-stepping for ice sheet models

- switch to **implicit time-stepping** for unconditional stability?
  - each step is a **free-boundary** NCP-coupled-to-Stokes problem
  - parameterize cost of these solves as $O(m^{1+\beta})$
- need q model updates per year to integrate climate influences

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(*diffusive*)	Stokes	$O\left(\frac{D\,L^{2+2\alpha}}{h^{4+2\alpha}}\right) = O\left(\frac{D}{L^2} m^{2+\alpha}\right)$
implicit		$O\left(\frac{q\,L^{2+2\beta}}{h^{2+2\beta}}\right) = O\left(q\, m^{1+\beta}\right)$

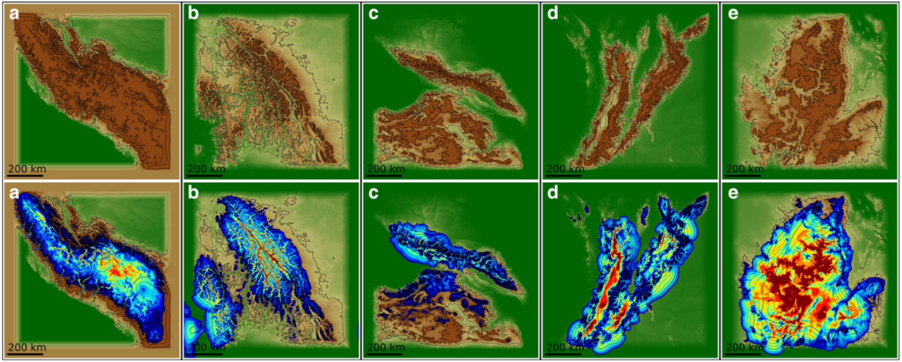
- goal for optimists: implicit time-stepping *and* build a $\beta \approx 0$ NCP-coupled-to-Stokes solver for the time step problems

existing implicit models?

- no convincing NCP-coupled-to-Stokes (free-boundary) solvers exist yet
  - Wirbel & Jarosch (2020) is an important attempt . . .
- the Bueler (2016) implicit (free-boundary) SIA solver scales badly: $\beta = 0.8$

Outline

- apply **machine learning**
- run non-scalable ice sheet models on many hypothetical/real ice sheets, and train ML **emulator** on results
  - supervised learning of physically-based model results
  - compute map on CPUs, then learn & evaluate map on GPUs
  - convincing demo in Jouvet et al. (2021)



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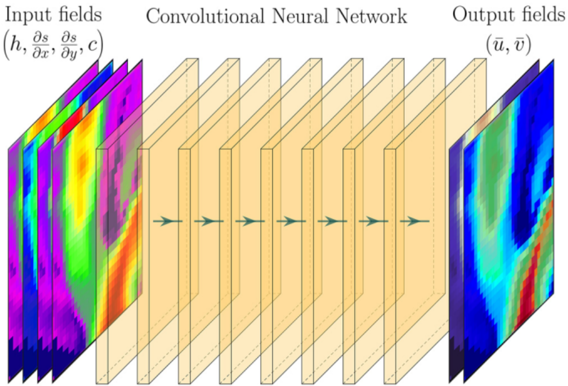

approach 2: semi-coupled time-stepping

- idea from Löfgren et al. (2022)
  - earlier use in mantle/crust simulations (Kaus et al. 2010)
- *idea.* remain explicit, but **modify the Stokes problem to "see" the updated (extrapolated) surface**
- that is, modify the Stokes problem at time t^ℓ by adding body force terms corresponding to the updated-surface icy domain

$$\int_{\Lambda^\ell} 2\nu D\mathbf{u} : D\mathbf{v} - \int_{\Lambda^\ell} p\nabla \cdot \mathbf{v} = \int_{\Lambda^\ell} \mathbf{f} \cdot \mathbf{v}$$

$$\int_{\Lambda^\ell} q\nabla \cdot \mathbf{u} = 0$$

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$$\int_{\Lambda^\ell} 2\nu D\mathbf{u} : D\mathbf{v} - \int_{\Lambda^\ell} p \nabla\cdot\mathbf{v} \qquad = \int_{\Lambda^\ell} \mathbf{f}\cdot\mathbf{v} + \Delta t \int_{\Gamma_s^\ell} a\,(\mathbf{f}\cdot\mathbf{v})\,dx$$

$$-\Delta t \int_{\Gamma_s^\ell} (\mathbf{u}\cdot\mathbf{n}_{s^\ell})(\mathbf{f}\cdot\mathbf{v})\,dx$$

$$\int_{\Lambda^\ell} q\nabla\cdot\mathbf{u} = 0$$

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approach 3: multilevel NCP-coupled-to-Stokes solvers

- direct attack on the problem seems to require a **multilevel** solver for **variational inequalities** (VIs)
- but in the **non-local residual case** $\leftarrow$ *yesterday's seminar*
  - this seems not to exist
  - the **smoother** must reduce a residual formed from surface-motion term $\Phi(s) = -\mathbf{u}|_s \cdot \mathbf{n}_s$ (from a scalable Stokes solver)
- near-optimal multilevel solvers exist for simpler VI problems

Outline

- ice sheet models solve a multi-scale, irregular-data problem with hard-to-observe boundary conditions
  - there are **no easy or magic techniques** for performance
- current-technology ice sheet models mostly use **explicit** time stepping, **non-optimal** stress-balance solvers, and **shallow** assumptions
  - progress is being made in all of these areas
- **coming soon** from current research:
  1. machine learning emulators (Jouvet et al. 2021)
  2. semi-coupled time stepping (Löfgren et al. 2022)
  3. scalable Stokes solvers (Isaac et al. 2015)
- scalable solvers for implicit-step NCP-coupled-to-Stokes models, which would seem to be the recommended numerical design, require **multilevel solvers for non-local variational inequalities**

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- E. Bueler (2016). *Stable finite volume element schemes for the shallow-ice approximation*, J. Glaciol. 62 (232), 230–242, 10.1017/jog.2015.3
- E. Bueler (2021). *Conservation laws for free-boundary fluid layers*, SIAM J. Appl. Math. 81 (5), 2007–2032, 10.1137/20M135217X
- E. Bueler (2022). *Performance analysis of high-resolution ice-sheet simulations*, J. Glaciol., 10.1017/jog.2022.113
- T. Isaac, G. Stadler, & O. Ghattas (2015). *Solution of nonlinear Stokes equations discretized by high-order finite elements on nonconforming and anisotropic meshes, with application to ice sheet dynamics*, SIAM J. Sci. Comput. 37 (6), B804–B833, 10.1137/140974407
- G. Jouvet & E. Bueler (2012). *Steady, shallow ice sheets as obstacle problems: well-posedness and finite element approximation*, SIAM J. Appl. Math. 72 (4), 1292–1314, 10.1137/110856654
- G. Jouvet, G. Cordonnier, B. Kim, M. Lüthi, A. Vieli, A. Aschwanden (2021). *Deep learning speeds up ice flow modelling by several orders of magnitude*, J. Glaciol. 10.1017/jog.2021.120
- A. Löfgren, J. Ahlkrona, & C. Helanow (2022). *Increasing stable time-step sizes of the free-surface problem arising in ice-sheet simulations*, J. Comput. Phys. X 16, 10.1016/j.jcpx.2022.100114