Worksheet: Decimal expansions of real numbers.

The goals here are *(1)* recall the precise meaning of decimal expansions, and *(2)* initiate the idea of completeness. Here we see, from decimal expansions, why all positive real numbers are "least upper bounds" for sets of rational numbers.

1. Consider the irrational real number $e = 2.718281828459045\ldots$. To confirm the meaning of this expansion, fill in the numerators:

$$e = 2 + \frac{\boxed{}}{10} + \frac{\boxed{}}{100} + \frac{\boxed{}}{1000} + \frac{\boxed{}}{10^4} + \frac{\boxed{}}{10^5} + \ldots$$

2. The terminating ("finite") decimal approximations of e are *rational numbers*, that is, fractions of natural numbers. Fill in numerators; there is no need to write in lowest terms:

$$2.70000000000\ldots = \frac{27}{10} \qquad 2.71820000000\ldots = \frac{\boxed{}}{10^4}$$

$$2.71000000000\ldots = \frac{\boxed{}}{100} \qquad 2.71828000000\ldots = \frac{\boxed{}}{10^5}$$

$$2.71800000000\ldots = \frac{\boxed{}}{1000} \qquad \vdots$$

3. Of course, the fractions in item **2** above are each *less than* e:

$$\begin{aligned} 2.70000000000\ldots &< e = 2.718281828459045\ldots \\ 2.71000000000\ldots &< e \\ 2.71800000000\ldots &< e \\ 2.71820000000\ldots &< e \\ &\vdots \end{aligned}$$

(*Nothing to fill in here. But please make sure you agree!*)

4. The main idea. Write an infinite set of rational numbers for which e is an *upper bound*, and for which no number smaller than e is an upper bound:

$$\left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

5. The number e is usually defined, for example during calculus II, as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

This gives another set of rational numbers which are less than e but for which no number less than e is also an upper bound. That is, use the partial sums; you may use summation notation:

$$\left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

6. In **4** and **5** it says "set of rational numbers". Of course, they are also *sequences* of rational numbers. What is the difference? Write a sentence or two.

7. Write a sequence (a_n) for **4**, and then a sequence (b_n) for **5**. For the latter you may use partial sum notation.

8. Do the same as in **4**, but for the irrational square root of 2, i.e. $\sqrt{2} = 1.4142135623730950488\ldots$.

$$\Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$$

9. Suppose you do not "know" the decimal expansion for $\sqrt{2}$. Describe in words how you might construct a specific set S of rational numbers for which $y = \sqrt{2}$ is an upper bound, and for which no smaller number is an upper bound. (*There are many correct answers.*)