

Problem 60. Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A . Assume c is a limit point of A . If $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} h(x) = L$ then $\lim_{x \rightarrow c} g(x) = L$.

Proof. □

Problem 61. If $h : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function then the set $K = \{x \in \mathbb{R} : h(x) = 0\}$ is closed.

Proof. □

Problem 62. If c is an isolated point of $A \subset \mathbb{R}$, and if $f : A \rightarrow \mathbb{R}$ is a function, then f is continuous at c .

Proof. □

Problem 63. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = \sqrt[3]{x}$ is continuous.

Proof. □

Problem 64. Dirichlet's function from Section 4.1, namely

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any $c \in \mathbb{R}$.

Proof. □

Problem 65. The function

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sqrt{|x|} \cos(1/x) & \text{otherwise,} \end{cases}$$

shown in the figure below, is continuous at zero.

Proof. □

Problem 66. Thomae's function from Section 4.1, namely

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } x = \pm m/n \text{ in lowest terms, with } n > 0, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous at any rational point $c \in \mathbb{Q}$.

Proof. □

Problem 67. Suppose $f : A \rightarrow \mathbb{R}$ is continuous at $c \in A$. Suppose that $g : B \rightarrow \mathbb{R}$ has a domain satisfying $f(A) \subset B$, and that g is continuous at $f(c)$. Let

$$h(x) = (g \circ f)(x) = g(f(x))$$

be the composition of functions. Then h is continuous at c .

Proof. □