Problem 51. *If $K \subset \mathbb{R}$ is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.*

Proof. □

Problem 52. *What of the following sets are compact? For those that are compact, give a brief justification. For those that are not, show how the book's definition of compact (Definition 3.3.1) breaks down. That is, give an example sequence in the set that does not contain a subsequence which converges to a point in the set.*

(a) $\mathbb{N}$

(b) $\mathbb{Q} \cap [0, 1]$

(c) The Cantor set C

(d) $\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \, : \, n \in \mathbb{N}\right\}$

(e) $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \ldots\right\}$

Problem 53. *If a set $K \subset \mathbb{R}$ is closed and bounded, then it is compact.*

Proof. □

Problem 54. *Decide whether the following propositions are true or false. If the claim is valid, supply a short proof. If the claim is false, provide a counterexample.*

(a) The arbitrary union of compact sets is compact.

(b) The arbitrary intersection of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ is a nested sequence of nonempty closed sets then the intersection $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty.

Problem 55. *Let A and B be nonempty subsets of $\mathbb{R}$. If there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.*

Proof. □

Problem 56. *A set $E \subset \mathbb{R}$ is* totally disconnected *if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$ and $y \in B$ and $E = A \cup B$.*

The Cantor set C is totally disconnected.

Proof. ☐

Problem 57. *For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge. Note that $[[x]]$ denotes the greatest integer which is less than or equal to x.*

(a) $\lim_{x \to 3} 5x - 6 = 9$, where $\epsilon = 1$

(b) $\lim_{x \to 4} \sqrt{x} = 2$, where $\epsilon = 0.5$

(c) $\lim_{x \to \pi} [[x]] = 3$, where $\epsilon = 0.5$

Problem 58. *Use the definition of functional limit in the textbook (Definition 4.2.1) to prove the following limit statements.*

(a) $\lim_{x \to 2} 3x + 4 = 10$

Proof. ☐

(b) $\lim_{x \to 2} x^2 + x - 1 = 5$

Proof. ☐

(c) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$

Proof. ☐

(d) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

Proof. ☐

Problem 59. *Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on A. Assume c is a limit point of A. If $\lim_{x \to c} g(x) = 0$ then $\lim_{x \to c} f(x)g(x) = 0$.*

Proof. ☐