with page number "solutions"

Review Guide for in-class Midterm Exam on Friday, 18 October 2024

The in-class Midterm Exam will cover the sections listed below from Griva, Nash, Sofer, *Linear and Nonlinear Optimization*, 2nd ed., 2009. It will only cover topics that have appeared on homework and in lecture. You are *not* responsible for any material in Chapters 6, 7, 8, 10, 12, 13, 14, 15, or 16.

The Exam is *closed book, no internet, and no calculator*. You may bring your own notes on *half* of a single sheet of letter-sized paper, but you can use both sides of that half sheet.

The problems will be of these types: state definitions, state or prove certain theorems (see below), prove or show propositions which follow directly from the definitions or from known facts, give examples with certain properties, describe or sketch concepts, or compute a step or two of certain algorithms.

I encourage you to get together with other students and work through this Review Guide. Be honest with yourself about what you do and don't know, and talk it through and learn!

The list of Definitions below has blanks for some page numbers. Please use it as a "take-home worksheet;" you can study by finding the page where the term is defined. On Wednesday 10/26 I will post "solutions" for these page numbers at the website.

Sections. You are responsible for the following material in these Sections:

- 1.1–1.6
- 2.1–2.6
- 3.1–3.3 (but *not* 3.3.2–3.3.4)
- 4.1–4.4
- 5.1–5.2 (but *not* 5.2.3, 5.2.4, or any other material on tableaus)
- Appendices A.3–A.5, A.7.1
- Appendices B.1–B.7

Definitions. Be able to define the term (word or phrase). Understand and/or use the term correctly. Be able to prove things that follow immediately from the definition:

- *transpose* $A^\top$ of an $m \times n$ matrix A p. ?
- *symmetric* matrix p. 661
- *nonsingular* matrix p. 665
- *positive definite* matrix p. 677
- *gradient* $\nabla f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ p. 694
- *Hessian* $\nabla^2 f(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ p. 694
- *Jacobian* $\nabla f(x)^\top$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ p. 695
- *feasible set* $S \subseteq \mathbb{R}^n$, e.g. as defined by constraints on page 43 p. 4, 44
- *feasible point* p. 44
- *active* constraint at $x \in S$ p. 44
- *inactive* inequality constraint at $x \in S$ p. 44

- *global minimizer* of f in S p. 45
- *strict global minimizer* p. 45
- *local minimizer* of f in S p. 46
- *strict local minimizer* p. 46
- *convex* set S p. 48
- *convex* function f on a convex set S p. 48
- *strictly convex* function p. 49
- *convex optimization problem* p. 49
- *convex combination* of a finite set of points in $\mathbb{R}^n$ p. 50
- *search direction* in an optimization algorithm p. 55
- *step length* p. 55
- *descent direction* p. 56
- *feasible direction* p. 78
- *feasible descent direction* p. 58
- *null space* of an $m \times n$ matrix A p. 82
- *range space* of A (or $A^\top$) p. 82
- *null space matrix* of A p. 83
- *standard form* of a linear programming problem (l.p.p.) p. 101
- *free variable* p. 102
- *slack variable* p. 102
- *excess variable* p. 102
- *extreme point* (or *vertex*) of a convex set S p. 107
- *feasible solution* of a standard form l.p.p. p. ?
- *basic feasible solution* of a standard form l.p.p. p. 107
- *optimal basic feasible solution* of a standard form l.p.p. p. 107
- *unbounded direction* of a standard form l.p.p. p. 112

Algorithms and Methods. (updated) Be able to state the algorithm or method. (For some, a pseudocode is an appropriate style.) Be able to apply/execute it in simple cases.

- General Optimization Algorithm II p. 55
- rules for converting LP problems to standard form (be able to *apply*) pp. 101–102
- simplex method for a LP problem in standard form p. 131

Theorems. (updated) Understand these theorems, and be able to use them as facts. Be able to illustrate with an example or a sketch. Be able to prove those that say so.

- Theorem 2.1 (global solutions of convex problems) ***be able to prove*** p. 49
- characterizations of convexity: p. 51
  - f is convex if it is above its tangent planes: $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$
  - f is strictly convex if its Hessian $\nabla^2 f(x)$ is positive definite for all x
- Taylor series, and Taylor's theorem with remainder, in one dimension pp. 64–65
- Taylor series, and Taylor's theorem with remainder, in $\mathbb{R}^n$, to $O(\|p\|^2)$ pp. 64–65
- Theorem 4.4 (extreme point $\iff$ basic feasible solution) p. 110
- Theorem 4.6 (representation of x as convex combination of b.f.s.) p. 120
- Theorem 4.7 (x optimal $\implies$ there is optimal b.f.s.) p. 121

Theorem about feasible directions for linear constraints. (*Ideas from Section 3.1 are stated separately here.*)

Understand the following things, be able to prove them, and be able to use them as facts. Be able to illustrate with an example or a sketch.

- notation for constraints:
$$\mathcal{E} \text{ is index set for } a_i^\top x = b_i, \qquad \mathcal{I} \text{ is index set for } a_i^\top x \geq b_i$$
  and $\hat{\mathcal{I}}$ denotes active inequality constraints at a given point $\bar{x}$
- p is a feasible direction at feasible point $\bar{x}$ if and only if p. 80
$$a_i^\top p = 0 \text{ for } i \in \mathcal{E} \qquad \text{and} \qquad a_i^\top p \geq 0 \text{ for } i \in \hat{\mathcal{I}}$$
- given a feasible direction p, only the inactive inequality constraints are relevant when determining an upper bound on step length α p. 81
- if $a_i^\top p \geq 0$ for all inactive inequality constraints then an arbitrarily-large step can be taken while staying feasible; there is no upper bound on α p. 81
- otherwise the maximum allowed step length arises from the *ratio test*: p. 81
$$\bar{\alpha} = \min \left\{ \frac{(a_i^\top \bar{x} - b)}{-a_i^\top p} \; : \; i \text{ is inactive at } \bar{x} \text{ and } a_i^\top p < 0 \right\}$$