"solutions" (i.e. my version)

Worksheet: Convexity proofs

Over the course of the semester there will be proofs on various topics, but also many concrete calculations to do and programs to write. For now, a bit of practice with the logic of convexity, and more examples of how to write proofs, might make things easier. I won't expect any particular proof style when grading homework and exams, but it might give confidence to have a target style.

A. (Same as Exercise 3.4 in Section 2.3.) Prove that a function f is concave if and only if $-f$ is convex.

Statement. Suppose f is a function on a convex set S and $g=-f$. Then f is concave if and only if g is convex.

Proof. ($\Rightarrow$) Suppose f is concave. Let $x,y \in S$ and $\alpha \in [0,1]$. Then

$$\begin{aligned} g(\alpha x+(1-\alpha)y) &= -f(\alpha x+(1-\alpha)y) \\ &\ge -(\alpha f(x)+(1-\alpha)f(y)) \\ &= \alpha g(x)+(1-\alpha)g(y), \end{aligned}$$

so g is concave. (Note the minus reverses the inequality.)

($\Leftarrow$) Suppose $g=-f$ is concave. Let $x,y\in S$ and $\alpha\in[0,1]$. Then

$$\begin{aligned} f(\alpha x+(1-\alpha)y) &= -g(\alpha x+(1-\alpha)y) \\ &\le -(\alpha g(x)+(1-\alpha)g(y)) = \alpha f(x)+(1-\alpha)f(y) \end{aligned}$$

so f is convex. $\square$

B. (Compare Exercise 3.13 in Section 2.3.) Prove that if g is concave then $S = \left\{x : g(x) \geq 0\right\}$ is convex.

Statement. Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is concave. Then $S = \{x : g(x) \geq 0\}$ is convex.

Proof. Let $x, y \in S$ and $\alpha \in [0,1]$. By definition of S, $g(x) \geq 0$ and $g(y) \geq 0$. Thus

$$g(\alpha x + (1-\alpha)y) \overset{\circledast}{\geq} \alpha g(x) + (1-\alpha) g(y)$$
$$\overset{\dagger}{\geq} \alpha \cdot 0 + (1-\alpha)\cdot 0 = 0.$$

(Step $\circledast$ is because g is concave. Step $\dagger$ is because $\alpha \geq 0$ and $(1-\alpha) \geq 0$.)

Thus $z = \alpha x + (1-\alpha) y$ is in S because $g(z) \geq 0$.

□