# MATH 661 Optimization

Monday 16 Sept.

- Assignment #2 due Wednesday 18 September
- Assignment #3 due Monday 23 September
  make sure you have version 2
- today:

special, short asynchronous lecture on Taylor series in n variables (section 2.6)

# Taylor series (etc.) in 1 variable

$f: \mathbb{R} \to \mathbb{R}$ or $f: I \subset \mathbb{R} \to \mathbb{R}$

**given:** function $f(x)$ and basepoint $x_0 \in \mathbb{R}$

($f$ needs to be defined and differentiable on an interval around $x_0$)

each in $\mathbb{R}$

**Taylor series:** $f(x_0+p) = f(x_0) + f'(x_0)p + \frac{1}{2}f''(x_0)p^2 + \dots + \frac{1}{k!}f^{(k)}(x_0)p^k + \dots$

**kth Taylor polynomial:** $q_k(p) = f(x_0) + f'(x_0)p + \frac{1}{2}f''(x_0)p^2 + \dots + \frac{1}{k!}f^{(k)}(x_0)p^k$

[**Casual:** $f(x_0+p) \approx f(x_0) + f'(x_0)p + \dots + \frac{1}{k!}f^{(k)}(x_0)p^k$]

Ex: find Taylor series and 4th Taylor polynomial for $f(x) = \ln x$ using base point $x_0 = 1$

Soln:

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = +2x^{-3}$

$f^{(4)}(x) = -3\cdot 2x^{-4}$

$f^{(5)}(x) = +4\cdot 3\cdot 2x^{-5}$

$\vdots$

$k \geq 1$: $f^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k}$

$$f(1+p) = 0 + 1\cdot p + \frac{1}{2}\cdot(-1)p^2 + \frac{1}{3!}(+1)2p^3 + \frac{1}{4!}(-1)3!\,p^4 + \frac{1}{5!}(+1)4!\,p^4 + \ldots$$

$$= p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \ldots (-1)^{k-1}\frac{p^k}{k} + \ldots$$

So:

$$q_4(p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4}$$

ideas: ① the Taylor series may only converge if $p$ is not too large ("radius of convergence")

② the Taylor series may not equal the original function

Ex of ①: $\ln(1+p) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{p^j}{j}$ from last slide

has $R=1$
radius

Ex of ②:

$$f(x) = \begin{cases} 0, & x=0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}, \quad x_0 = 0$$

don't worry about it here

then the series is $0+0p+0p^2+0p^3+\dots \neq f(p)$

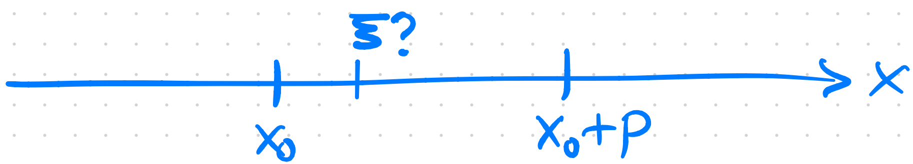
- observe where I used "=" versus "≈" on previous slides ... be careful this way please!

- sometimes you want Taylor's theorem:

$$f(x_0+p) = f(x_0) + f'(x_0)p + \dots + \frac{1}{k!} f^{(k)}(x_0) p^k + \underbrace{\frac{1}{(k+1)!} f^{(k+1)}(\xi)\, p^{k+1}}_{\text{remainder term}}$$

yes, "=" here

where $\xi$ is some number between $x_0$ and $x_0+p$



- in optimization the uses of Taylor stuff are often just the linear or quadratic approximations:

$$f(x_0+p) \approx \underbrace{f(x_0) + f'(x_0)p}_{=q_1(p)} \quad \text{linear approx.}$$

$$f(x_0+p) \approx \underbrace{f(x_0) + f'(x_0)\,p + \tfrac{1}{2} f''(x_0)\,p^2}_{q_2(p)} \quad \text{quadratic approx.}$$

- <u>**but**</u> we want these in $\mathbb{R}^n$!

# Taylor series in n variables, but only out to 2nd order

$S \subset \mathbb{R}^n$ open

$f: S \to \mathbb{R}$ is continuous, and all its derivatives are continuous

$x_0 \in S$

then for $p \in \mathbb{R}^n$:

$$f(x_0+p) = f(x_0) + \nabla f(x_0)^T p + \frac{1}{2} p^T \nabla^2 f(x_0) p$$

$$+ \cdots$$

in n variables it is not clear (initially) what this is

## We don't care about higher order:

<u>all</u> our optimization uses of Taylor in n variables will be

$$\underset{\text{linear approx.}}{f(x_0+p) \approx f(x_0) + \overset{\text{gradient}}{\nabla f(x_0)}^T p}$$

or

$$\underset{\text{quadratic approx}}{f(x_0+p) \approx f(x_0) + \nabla f(x_0)^T p + \frac{1}{2} p^T \underset{\text{Hessian}}{\nabla^2 f(x_0)} p}$$

Ex: Find the first 3 terms of the Taylor series for

$$f(x) = 3x_1^4 - x_1x_2 + 5x_1x_2^2 + 2$$

at the point $x_0 = (1, 1)^T$.

Evaluate the series for $p = (0.1, -0.1)^T$ and compare with the value of $f(x_0 + p)$.

asked just like Exercise 6.4

soln corrected on next two slides

Solution: $f(x) = 3x_1^4 - x_1 x_2 + 5x_1 x_2^2 + 2$ $\quad n=2$

$$\nabla f(x) = \begin{bmatrix} 12x_1^3 - x_2 + 5x_2^2 \\ -x_1 + 10x_1 x_2 \end{bmatrix}$$

corrected

$$\nabla^2 f(x) = \left[\begin{array}{c|c} 36x_1^2 & -1 + 10x_2 \\ \hline -1 + 10x_2 & 10x_1 \end{array}\right]$$

$x_0 = (1,1)^T$ so:

$$f(x_0+p) \approx \underbrace{9}_{f(x_0)} + \underbrace{\begin{bmatrix} 16 & 9 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{\nabla f(x_0)^T p} + \underbrace{\frac{1}{2} \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 36 & 9 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{\frac{1}{2} p^T \nabla^2 f(x_0) p}$$

$$= 9 + 16p_1 + 9p_2 + 18p_1^2 + 9p_1p_2 + 5p_2^2$$

quadratic in $p \in \mathbb{R}^2$

and with $p = (0.1, -0.1)^T$:

$$q_2(p) = 9.84$$

$$f(x_0 + p) = 9.8573$$

corrected

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Matlab in support of above:

```
>> f = @(x) 3*x(1)^4 - x(1)*x(2) +5*x(1)*x(2)^2 +2;
>> q = @(p) 9+16*p(1)+9*p(2)+ 18*p(1)^2+9*p(1)*p(2)
            + 5*p(2)^2;
>> f([1.1,0.9]) = 9.8573
>> q([0.1,-0.1]) = 9.84
```

## Our main use of Taylor ideas:

- consider a smooth, unconstrained, generally <u>not</u> convex optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

- suppose $x_j \in \mathbb{R}^n$ is our current iterate in some optimization algorithm

- Then

$$g(p) = f(x_j) + \nabla f(x_j)^T p$$

is our "linear model of $f$ near $x_j$"

and

$$h(p) = f(x_j) + \nabla f(x_j)^T p + \frac{1}{2} p^T \nabla^2 f(x_j) p$$

is our "quadratic model of $f$ near $x_j$"

- the next step in the optimization algorithm is

$$x_{j+1} = x_j + p_*$$

compute this as

$$\min_p g(p)$$

or $$\min_p h(p)$$

possibly subject to "not too far" constraints