Name: SOLUTIONS

Math 661 Optimization (Bueler)

Friday, 18 October 2024

Midterm Exam

65 minutes. No book. No electronics or internet. 1/2 **sheet of notes allowed.**
(100 *points possible*)

1. (*4 pts*) Given a matrix $A \in \mathbb{R}^{n\times n}$, define what it means for A to be *positive definite.*

A is positive definite if $x^T A x > 0$ for all nonzero $(x \neq 0)$ vectors $x \in \mathbb{R}^n$

2. **(a)** (*4 pts*) Define *convex set* (for a subset $S \subset \mathbb{R}^n$).

S is convex if for all $x, y \in S$, and all $0 \le \alpha \le 1$, $\alpha x + (1-\alpha) y \in S$

(b) (*4 pts*) Define *convex function* (for a real-valued function f defined on a convex set $S \subset \mathbb{R}^n$).

f is convex if, for all $x, y \in S$ and all $0 \le \alpha \le 1$,

$$f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y)$$

(c) (*4 pts*) For a convex set $S \subset \mathbb{R}^n$, define what it means for $x \in S$ to be an *extreme point.*

$x \in S$ is an extreme point if there does not exist $y, z \in S$, with $x \neq y$ and $x \neq z$, and $0 < \alpha < 1$, so that $x = \alpha y + (1-\alpha) z$

3. **(a)** (*4 pts*) State the *standard form* of a linear programming problem.

$$\begin{aligned} &\text{minimize} && z = c^T x \\ &\text{subject to} && Ax = b \\ & && x \geq 0 \end{aligned} \qquad \left[\begin{array}{l} c \in \mathbb{R}^n \ (x \in \mathbb{R}^n) \\ A \in \mathbb{R}^{m \times n} \\ b \in \mathbb{R}^m \end{array}\right]$$

with $b \geq 0$

(b) (*4 pts*) For a problem in standard form, define *basic solution.*

x is a <u>basic solution</u> if $Ax = b$ and if the nonzero entries of x correspond to linearly-independent columns of A

4. Let $f(x) = 2x_3x_2 + x_3^2 - x_2 - 2x_1^2$ for $x \in \mathbb{R}^3$.

(a) (*6 pts*) Compute the gradient and Hessian of f at $\tilde{x} = (-1, 1, 1)^\top \in \mathbb{R}^3$.

$$\nabla f(x) = \begin{bmatrix} -4x_1 \\ 2x_3 - 1 \\ 2x_2 + 2x_3 \end{bmatrix} \implies \nabla f(\tilde{x}) = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \nabla^2 f(\tilde{x})$$

(b) (*5 pts*) Is $p = (-2, 1, 0)^\top$ a descent direction for f at $\tilde{x}$ from part **(a)**?

The question is whether $\nabla f(\tilde{x})^T p < 0$.

but

$$\nabla f(\tilde{x})^T p = \begin{bmatrix} 4 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -8 + 1 + 0 = -7$$

So <u>yes</u>

5. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \exp(x_1^4 + x_2^2) - x_1^4 + \sin(x_1 x_2 x_3) \\ \text{subject to} & 2x_1 - 2x_2 + x_3 = -1 \quad (1) \\ & x_1 + 4x_2 \geq -5 \quad (2) \\ & x_2 \geq -1 \quad (3) \end{array}$$

(a) (*5 pts*) Is $x = (2, 0, -5)^\top$ feasible?

(1) $2 \cdot 2 - 0 + (-5) = 4 - 5 = -1$ ✓

(2) $2 + 0 \geq -5$ ✓

(3) $0 \geq -1$ ✓

$\}$ yes

(b) (*5 pts*) Considering all of the constraints, both equality and inequality, which are active and which are inactive at the feasible point $\tilde{x} = (0, -1, -3)^\top$?

(1) $0 + 2 - 3 = -1$ active

(2) $0 - 4 \geq -5$ inactive

(3) $-1 \geq -1$ active

Extra Credit A. (*3 pts*) For $x \in \mathbb{R}^n$, completely solve the standard-form linear programming problem in which there are no equality constraints:

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & x \geq 0 \end{array}$$

(*Hint.* Don't do simplex (or other) method. Please think about it. Consider all cases for c.)

Case 1: any entry of c is negative $(c_i < 0)$ then there is no solution because $x = \alpha e_i \geq 0$ gives $z = c^T x = \alpha c_i$; is arbitrarily negative

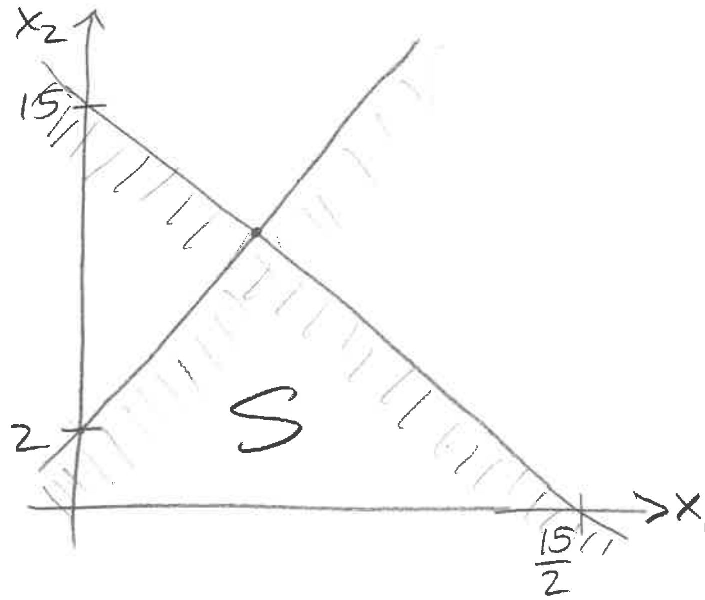
Case 2: $c > 0$. then $x = 0$ is the unique solution; it is a local (hence global by convexity) minimizer because there are no feasible descent directions

Case 3: $c \geq 0$ but some $c_i = 0$. multiple minimizers (with any $x_i \geq 0$)

6. (a) (*5 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = 3x_1 - 8x_2 \\ \text{subject to} & 2x_1 + x_2 \le 15 \\ & 3x_1 - x_2 \ge -2 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

$-3x_1 + x_2 \le 2$



(b) (*5 pts*) Convert the problem in part (a) to standard form.

$$\begin{array}{ll} \min & z = 3x_1 - 8x_2 + 0x_3 + 0x_4 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 15 \\ & -3x_1 + x_2 + x_4 = 2 \\ & x \ge 0 \end{array}$$

7. (*4 pts*) Given a feasible set $S \in \mathbb{R}^n$, and a feasible point $\tilde{x} \in S$, define what it means for $p \in \mathbb{R}^n$ to be a *feasible direction*.

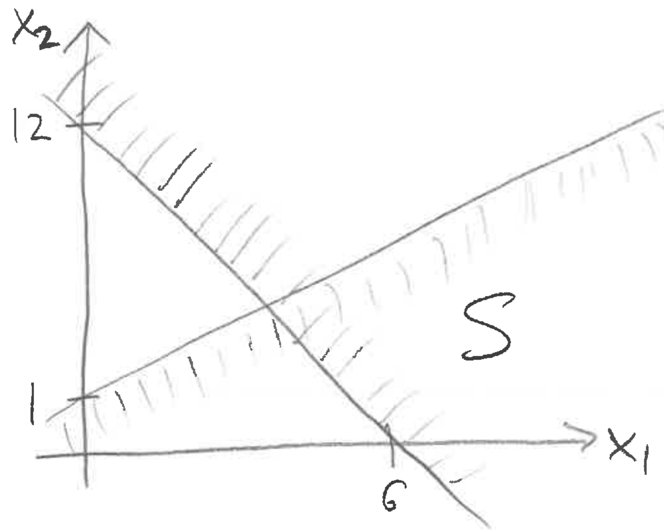
p is a feasible direction if there is $\varepsilon > 0$ so that $\tilde{x} + \alpha p \in S$ when $0 < \alpha < \varepsilon$.

8. **(a)** (*5 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 12 \\ & -x_1 + 3x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

$x_2 \geq 12 - 2x_1$

$x_2 \leq \frac{3+x_1}{3} = 1 + \frac{x_1}{3}$



(b) (*5 pts*) Convert the problem in part **(a)** to standard form.

$$\begin{array}{ll} \min & z = -x_1 - 2x_2 + 0x_3 + 0x_4 \\ \text{s.t.} & 2x_1 + x_2 - x_3 \quad = 12 \\ & -x_1 + 3x_2 \quad + x_4 = 3 \\ & x \geq 0 \end{array}$$

(c) (*5 pts*) At the feasible point $\tilde{x} = (6, 0)^\top$, for the original problem, notice that $p = (0, 1)^\top$ is a feasible direction. What is the maximum $\alpha > 0$ so that $x = \tilde{x} + \alpha p$ is feasible?

$x = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ \alpha \end{bmatrix}$. question is when 2nd constraint becomes active: $-x_1 + 3x_2 = 3$

$-6 + 3\alpha = 3$

$3\alpha = 9$

$\alpha = 3$

(d) (*5 pts*) Does this linear programming problem have a solution? Explain briefly.

no. in the form of part (a),

$d = (1, 0)^T$

is a direction of unboundedness, for example, and $z = x_1 + 2x_2$ is arbitrarily large in this direction

9. **(a)** (*5 pts*) Consider the standard form linear programming problem

$$\begin{array}{ll} \text{minimize} & z = -x_1 + x_2 + 0x_3 + 0x_4 \\ \text{subject to} & -x_1 + x_2 + x_3 \quad\quad = 3 \\ & 2x_1 + x_2 \quad\quad + x_4 = 4 \\ & x \geq 0 \end{array}$$

Find a basic feasible solution x with $x_1 = 0$ and $x_2 = 0$.

$$\left.\begin{array}{l} 0 + 0 + x_3 = 3 \\ 0 + 0 + x_4 = 4 \end{array}\right\} \longrightarrow x = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

(b) (*8 pts*) Let x be the basic feasible solution from part **(a)**. Use the template below to complete one iteration of the (revised) simplex method. **At the bottom**, fill in the basic and non-basic variables (indices) at the completion of this first iteration.

$\mathcal{B} = \{3, 4\}$, $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\quad \underline{Bx_B = b} \implies x_B = \hat{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\mathcal{N} = \{1, 2\}$, $\quad N = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $\quad c_N = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0$

$I\hat{A}_t = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

~~$\boxed{\hat{c}_N \geq 0\text{?: stop with optimum}}$~~ $\quad \hat{c}_N \xrightarrow{\text{index of}}_{\text{min}} t = \boxed{1} \rightarrow \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\boxed{\hat{A}_t \leq 0\text{?: stop, unbounded}}$ $\quad \left\{ \frac{\hat{b}_i}{\hat{a}_{i,t}} \right\} = \left\{ \frac{4}{2} \right\} \xrightarrow{\text{index of}}_{\text{min over } \hat{a}_{i,t} > 0} s = \boxed{4}$

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result: $\quad \mathcal{B} = \{1, 3\}$, $\quad \mathcal{N} = \{2, 4\}$

10. (*8 pts*) Prove the following theorem.

Theorem. *If x_* is a local minimizer of a convex optimization problem then x_* is also a global minimizer.*

Proof. Suppose $x_* \in S$ is a local minimizer but it is not a global minimizer. Then there is $y \in S$ so that $f(y) < f(x_*)$.

But S is convex so $z = \alpha y + (1-\alpha) x_*$ is in S. And f is convex so

$$f(z) \leq \alpha f(y) + (1-\alpha) f(x_*).$$

(These statements are true for $0 \leq \alpha \leq 1$.)

Choose $\alpha > 0$. Then

$$f(z) \leq \alpha f(y) + (1-\alpha) f(x_*) < \alpha f(x_*) + (1-\alpha) f(x_*)$$

so $f(z) < f(x_*)$. But $\|z - x_*\| = \alpha \|y - x_*\|$ can be chosen as small as desired. So x_* is not a local minimizer, a contradiction. $\square$

Extra Credit B. (*2 pts*) Suppose $A \in \mathbb{R}^{m \times n}$ has full row rank. Suppose that this orthogonal factorization has been done:

$$A^\top = QR. \qquad \Longrightarrow \quad A = R^T Q^T$$

(Here Q is square with orthonormal columns, and R is upper triangular.) Explain how to use this factorization to form a null space matrix Z for A.

$$\underset{n \times m}{\begin{bmatrix} A^T \end{bmatrix}} = \underset{n \times n}{\begin{bmatrix} Q \end{bmatrix}} \underset{n \times m}{\begin{bmatrix} r_{11} & r & \cdots \\ & \ddots & r \\ 0 & & r_m \\ \hline & 0 & \end{bmatrix}} = \begin{bmatrix} Q_1 & | & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ \hline 0 \end{bmatrix}$$

so $\boxed{Z = Q_2}$, an $n \times m$ matrix (because $AZ = AQ_2 = [R_1^T | 0] \begin{bmatrix} Q_1^T \\ \hline Q_2^T \end{bmatrix} [Q_2] = 0$)

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