Name: ______________________________

Math 661 Optimization (Bueler) Friday, 18 October 2024

Midterm Exam

65 minutes. No book. No electronics or internet. 1/2 **sheet of notes allowed.**

(100 *points possible*)

1. (*4 pts*) Given a matrix $A \in \mathbb{R}^{n\times n}$, define what it means for A to be *positive definite*.

2. **(a)** (*4 pts*) Define *convex set* (for a subset $S \subset \mathbb{R}^n$).

(b) (*4 pts*) Define *convex function* (for a real-valued function f defined on a convex set $S \subset \mathbb{R}^n$).

(c) (*4 pts*) For a convex set $S \subset \mathbb{R}^n$, define what it means for $x \in S$ to be an *extreme point*.

3. **(a)** (*4 pts*) State the *standard form* of a linear programming problem.

(b) (*4 pts*) For a problem in standard form, define *basic solution.*

4. Let $f(x) = 2x_3x_2 + x_3^2 - x_2 - 2x_1^2$ for $x \in \mathbb{R}^3$.

(a) (*6 pts*) Compute the gradient and Hessian of f at $\tilde{x} = (-1, 1, 1)^\top \in \mathbb{R}^3$.

(b) (*5 pts*) Is $p = (-2, 1, 0)^\top$ a descent direction for f at $\tilde{x}$ from part **(a)**?

5. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \exp(x_1^4 + x_2^2) - x_1^4 + \sin(x_1 x_2 x_3) \\ \text{subject to} & 2x_1 - 2x_2 + x_3 = -1 \\ & x_1 + 4x_2 \geq -5 \\ & x_2 \geq -1 \end{array}$$

(a) (*5 pts*) Is $x = (2, 0, -5)^\top$ feasible?

(b) (*5 pts*) Considering all of the constraints, both equality and inequality, which are active and which are inactive at the feasible point $\tilde{x} = (0, -1, -3)^\top$?

Extra Credit A. (*3 pts*) For $x \in \mathbb{R}^n$, completely solve the standard-form linear programming problem in which there are no equality constraints:

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & x \geq 0 \end{array}$$

(*Hint.* Don't do simplex (or other) method. Please think about it. Consider all cases for c.)

6. **(a)** (*5 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = 3x_1 - 8x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 15 \\ & 3x_1 - x_2 \geq -2 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(b) (*5 pts*) Convert the problem in part **(a)** to standard form.

7. (*4 pts*) Given a feasible set $S \in \mathbb{R}^n$, and a feasible point $\tilde{x} \in S$, define what it means for $p \in \mathbb{R}^n$ to be a *feasible direction.*

8. **(a)** (*5 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 12 \\ & -x_1 + 3x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(b) (*5 pts*) Convert the problem in part **(a)** to standard form.

(c) (*5 pts*) At the feasible point $\tilde{x} = (6,0)^\top$ for the original problem, notice that $p = (0,1)^\top$ is a feasible direction. What is the maximum $\alpha > 0$ so that $x = \tilde{x} + \alpha p$ is feasible?

(d) (*5 pts*) Does this linear programming problem have a solution? Explain briefly.

9. **(a)** (*5 pts*) Consider the standard form linear programming problem

$$\begin{array}{ll} \text{minimize} & z = -x_1 + x_2 \\ \text{subject to} & -x_1 + x_2 + x_3 = 3 \\ & 2x_1 + x_2 + x_4 = 4 \\ & x \geq 0 \end{array}$$

Find a basic feasible solution x with $x_1 = 0$ and $x_2 = 0$.

(b) (*8 pts*) Let x be the basic feasible solution from part **(a)**. Use the template below to complete one iteration of the (revised) simplex method. **At the bottom**, fill in the basic and non-basic variables (indices) at the completion of this first iteration.

$\mathcal{B} = \{ \quad \}$, $B = \begin{bmatrix} \quad \end{bmatrix}$, $c_B = \begin{bmatrix} \quad \end{bmatrix}$, $\underline{Bx_B = b} \implies x_B = \hat{b} = \begin{bmatrix} \quad \end{bmatrix}$

$\mathcal{N} = \{ \quad \}$, $N = \begin{bmatrix} \quad \end{bmatrix}$, $c_N = \begin{bmatrix} \quad \end{bmatrix}$

$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} \quad \end{bmatrix} \implies \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} \quad \end{bmatrix}$

$\boxed{\hat{c}_N \geq 0\text{?: stop with optimum}}$ $\hat{c}_N \xrightarrow[\text{min}]{\text{index of}}$ $t = \boxed{}$ $\rightarrow \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

$\boxed{\hat{A}_t \leq 0\text{?: stop, unbounded}}$ $\left\{ \frac{\hat{b}_i}{\hat{a}_{i,t}} \right\} = \{ \quad \}$ $\xrightarrow[\text{min over } \hat{a}_{i,t} > 0]{\text{index of}}$ $s = \boxed{}$

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result: $\mathcal{B} = \{ \quad \}$, $\mathcal{N} = \{ \quad \}$

10. (*8 pts*) Prove the following theorem.

Theorem. *If x_* is a local minimizer of a convex optimization problem then x_* is also a global minimizer.*

Proof.

Extra Credit B. (*2 pts*) Suppose $A \in \mathbb{R}^{m\times n}$ has full row rank. Suppose that this orthogonal factorization has been done:

$$A^\top = QR.$$

(Here Q is square with orthonormal columns, and R is upper triangular.) Explain how to use this factorization to form a null space matrix Z for A.

BLANK PAGE FOR SCRATCH WORK. CLEARLY-LABEL ANYTHING YOU WANT TO BE GRADED