Name:

Math 661 Optimization (Bueler)

Friday, 18 October 2024

Midterm Exam

65 minutes. No book. No electronics or internet. 1/2 sheet of notes allowed. (100 points possible)

1. (4 *pts*) Given a matrix $A \in \mathbb{R}^{n \times n}$, define what it means for A to be *positive definite*.

2. (a) (4 pts) Define convex set (for a subset $S \subset \mathbb{R}^n$).

(b) (4 pts) Define convex function (for a real-valued function f defined on a convex set $S \subset \mathbb{R}^n$).

(c) (4 pts) For a convex set $S \subset \mathbb{R}^n$, define what it means for $x \in S$ to be an *extreme point*.

3. (a) (4 pts) State the standard form of a linear programming problem.

(b) (4 pts) For a problem in standard form, define basic solution.

4. Let $f(x) = 2x_3x_2 + x_3^2 - x_2 - 2x_1^2$ for $x \in \mathbb{R}^3$. (a) (6 pts) Compute the gradient and Hessian of f at $\tilde{x} = (-1, 1, 1)^\top \in \mathbb{R}^3$.

(b) (5 pts) Is $p = (-2, 1, 0)^{\top}$ a descent direction for f at \tilde{x} from part (a)?

5. Consider the optimization problem

minimize
$$f(x) = \exp(x_1^4 + x_2^2) - x_1^4 + \sin(x_1 x_2 x_3)$$

subject to $2x_1 - 2x_2 + x_3 = -1$
 $x_1 + 4x_2 \ge -5$
 $x_2 \ge -1$

(a) (5 pts) Is $x = (2, 0, -5)^{\top}$ feasible?

(b) (5 pts) Considering all of the constraints, both equality and inequality, which are active and which are inactive at the feasible point $\tilde{x} = (0, -1, -3)^{\top}$?

Extra Credit A. (3 pts) For $x \in \mathbb{R}^n$, completely solve the standard-form linear programming problem in which there are no equality constraints:

 $\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & x \ge 0 \end{array}$

(*Hint.* Don't do simplex (or other) method. Please think about it. Consider all cases for c.)

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- 6. (a) (5 pts) Sketch the feasible set for the following linear programming problem:

minimize $z = 3x_1 - 8x_2$
subject to $2x_1 + x_2 \le 15$
 $3x_1 - x_2 \ge -2$
 $x_1 \ge 0, x_2 \ge 0$

(b) (5 pts) Convert the problem in part (a) to standard form.

7. (4 pts) Given a feasible set $S \in \mathbb{R}^n$, and a feasible point $\tilde{x} \in S$, define what it means for $p \in \mathbb{R}^n$ to be a *feasible direction*.

8. (a) (5 pts) Sketch the feasible set for the following linear programming problem:

maximize	$z = x_1 + 2x_2$
subject to	$2x_1 + x_2 \ge 12$
	$-x_1 + 3x_2 \le 3$
	$x_1 \ge 0, x_2 \ge 0$

(b) (5 pts) Convert the problem in part (a) to standard form.

(c) (5 pts) At the feasible point $\tilde{x} = (6, 0)^{\top}$ for the original problem, notice that $p = (0, 1)^{\top}$ is a feasible direction. What is the maximum $\alpha > 0$ so that $x = \tilde{x} + \alpha p$ is feasible?

(d) (5 pts) Does this linear programming problem have a solution? Explain briefly.

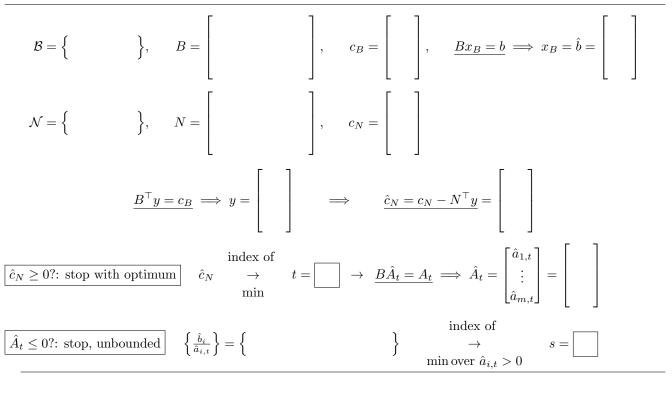
 $\mathbf{6}$

9. (a) (5 pts) Consider the standard form linear programming problem

minimize $z = -x_1 + x_2$
subject to $-x_1 + x_2 + x_3 = 3$
 $2x_1 + x_2 + x_4 = 4$
x > 0

Find a basic feasible solution x with $x_1 = 0$ and $x_2 = 0$.

(b) $(8 \ pts)$ Let x be the basic feasible solution from part (a). Use the template below to complete one iteration of the (revised) simplex method. At the bottom, fill in the basic and non-basic variables (indices) at the completion of this first iteration.



result: $\mathcal{B} = \{ \}, \quad \mathcal{N} = \{ \}$

10. (8 pts) Prove the following theorem.

Theorem. If x_* is a local minimizer of a convex optimization problem then x_* is also a global minimizer. Proof.

Extra Credit B. (2 *pts*) Suppose $A \in \mathbb{R}^{m \times n}$ has full row rank. Suppose that this orthogonal factorization has been done:

$$A^{+} = QR.$$

(Here Q is square with orthonormal columns, and R is upper triangular.) Explain how to use this factorization to form a null space matrix Z for A.

BLANK PAGE FOR SCRATCH WORK. CLEARLY-LABEL ANYTHING YOU WANT TO BE GRADED