Name: SOLUTIONS

Math 661 Optimization (Bueler) Friday, 28 October 2022

Midterm Exam

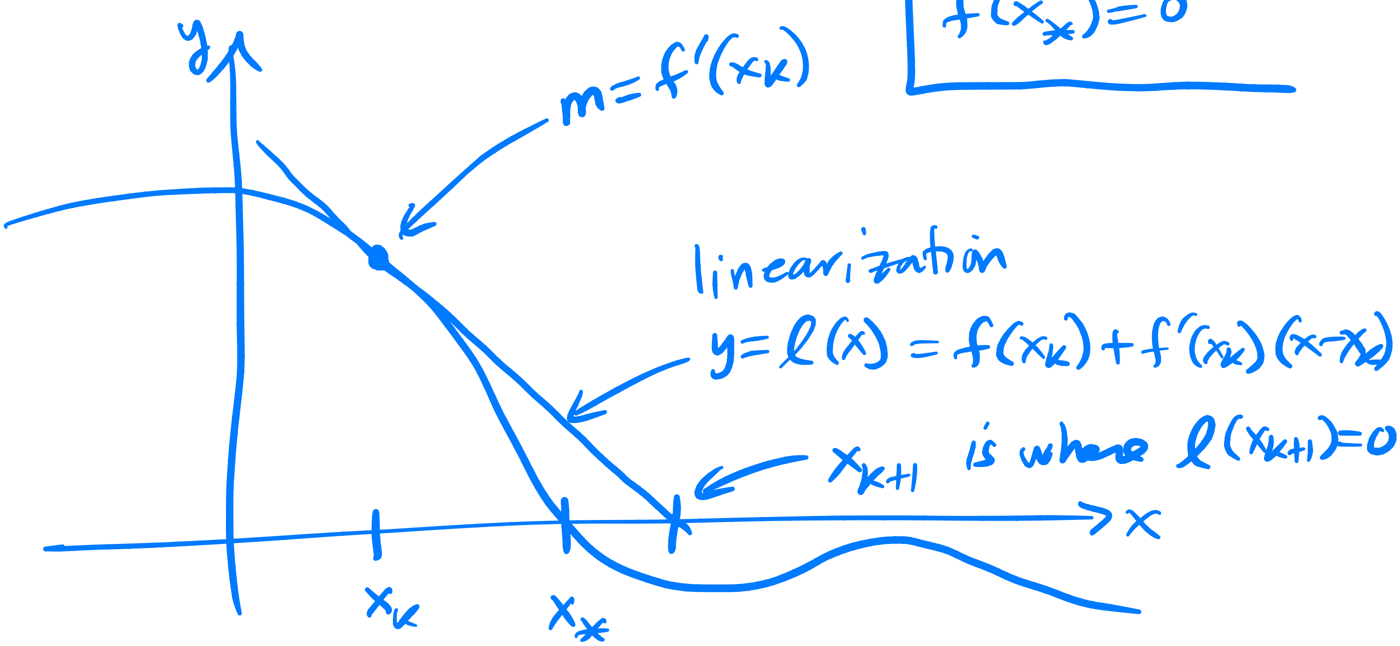
In class. No book. No calculator. 1/2 **sheet of notes allowed.**

(100 *points possible*)

1. Consider Newton's method to solve the scalar equation $f(x) = 0$.

(a) (*8 pts*) Draw and label a sketch of one step of Newton's method. In particular, your graph should show $y = f(x)$ as a generic curve, then an iterate x_k, and then show (graphically) how the next iterate x_{k+1} is determined.

$\boxed{f(x_*) = 0}$



(b) (*5 pts*) Do one step of Newton's method to solve the equation $x^3 - x + 1 = 0$, starting at $x_0 = 1$. That is, compute x_1.

$$\left.\begin{aligned} f(x) &= x^3 - x + 1 \\ f'(x) &= 3x^2 - 1 \end{aligned}\right\}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{1 - 1 + 1}{3 - 1}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

2. Let $f(x) = 4x_3x_2 + x_3^2 + x_2 - 2x_1^2$ for $x \in \mathbb{R}^3$.

(a) (*4 pts*) Compute the gradient and Hessian of f at $x_k = (-1, 1, 1)^\top \in \mathbb{R}^3$.

$$\nabla f(x) = \begin{bmatrix} -4x_1 \\ 4x_3 + 1 \\ 4x_2 + 2x_3 \end{bmatrix} \Bigg\} \Rightarrow \nabla f(x_k) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 2 \end{bmatrix} = \nabla^2 f(x_k)$$

(b) (*4 pts*) Does $f(x)$ have any stationary points? If so, find them.

yes

$\nabla f(x_*) = 0$

$$\Leftrightarrow \begin{array}{l} -4x_1 = 0 \\ 4x_3 + 1 = 0 \\ 4x_2 + 2x_3 = 0 \end{array} \Leftrightarrow \begin{array}{l} x_1 = 0 \\ x_2 = -\frac{1}{2}x_3 = \frac{1}{8} \\ x_3 = -1/4 \end{array} \Leftrightarrow x_* = \begin{bmatrix} 0 \\ 1/8 \\ -1/4 \end{bmatrix}$$

only stationary point

(c) (*4 pts*) Find all the local minima x_* of f, or explain why none exist. Justify your answer using appropriate 1st- or 2nd-order necessary or sufficient conditions.

$H = \nabla^2 f(x)$ is constant. it is not positive definite or positive semi-definite because $h_{11} = -4$.

So 2nd-order necessary condition shows (single) stationary point is not a local min. there are no local minima

(d) (*4 pts*) Is $p = (-2, 1, 0)^\top$ a descent direction for f at x_k from part **(a)**?

$$\nabla f(x_k)^\top p = [4, 5, 6]\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -8 + 5 + 0 = -3 < 0$$

So yes p is a descent direction

3. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \exp(x_1^4 + x_2^2) - x_1^4 + \sin(x_1 x_2 x_3) \\ \text{subject to} & 2x_1 - 2x_2 + x_3 = -1 \\ & x_1 + 4x_2 \geq -3 \\ & 7x_2 - 5x_3 \geq -1 \end{array}$$

(a) (*4 pts*) Is $x = (-2, 0, 3)^\top$ feasible?

$-4 - 0 + 3 = -1$ ✓

$-2 + 0 \geq -3$ ✓

$-0 - 15 \geq -1$ ✗

no (3rd constraint not satisfied)

(b) (*4 pts*) Considering both equality and inequality constraints, which constraints are active and which are inactive at $x = (1, 3, 3)^\top$?

x is feasible:

$2 - 6 + 3 = -1$ ✓

$1 + 12 > -3$ ✓

$21 - 15 > -1$ ✓

Since both inequalities are strict at x, only the equality constraint is active while inequality constraints are inactive

Extra Credit. (*3 pts*) For $x \in \mathbb{R}^n$, completely solve the standard-form linear programming problem when there are no equality constraints:

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & x \geq 0 \end{array}$$

Claims: ① if any entry in c is negative ($c_i < 0$) then the problem is unbounded $\therefore$ no optima [$c^\top x$ can be as negative as desired]

② if $c_i \geq 0$ for all i then $x = 0$ is an optimum; other optima where $x_i > 0$ if $c_i = 0$

③ if $c_i > 0$ for all i the $x = 0$ is the unique optimum

4. (*10 pts*) Consider general minimization problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & a_j^\top x = b_j \quad \text{for } j \in \mathcal{E} \\ & a_j^\top x \geq b_j \quad \text{for } j \in \mathcal{I} \end{array}$$

for given vectors $a_j \in \mathbb{R}^n$ and scalars b_j.

Suppose $\bar{x}$ is a point in the feasible set. Let $\hat{\mathcal{I}}$ be the set of indices $j \in \mathcal{I}$ where the inequality constraint $a_j^\top x \geq b_j$ is active at $\bar{x}$. Show that if $a_j^\top p = 0$ for all $j \in \mathcal{E}$, and if $a_j^\top p \geq 0$ for all $j \in \hat{\mathcal{I}}$, then p is a feasible direction.

<u>Proof:</u> Recall p is a feasible direction at $\bar{x}$ if $\bar{x} + \alpha p$ is feasible for all sufficiently small $\alpha > 0$. For $j \in \mathcal{E}$, $a_j^T(\bar{x} + \alpha p) = a_j^T \bar{x} + \alpha \cancel{a_j^T p}$

$$= b_j + \alpha \cdot 0 = b_j$$

so $\bar{x} + \alpha p$ satisfies equality constraints.

For $j \in \hat{\mathcal{I}}$, $a_j^T(\bar{x} + \alpha p) = a_j^T \bar{x} + \alpha a_j^T p \geq b_j + 0$

so $\hat{\mathcal{I}}$ inequality constraints still hold for any $\alpha > 0$.

For $j \in \mathcal{I} \setminus \hat{\mathcal{I}}$, $a_j^T(\bar{x} + \alpha p) = a_j^T \bar{x} + \alpha a_j^T p$

$> b_j + \alpha a_j^T p$. This value is nonnegative if either $\frac{-b_j}{a_j^T p} > 0$ or if $0 < \alpha < \frac{-b_j}{a_j^T p}$.

So all constraints hold if $\alpha > 0$ is sufficiently small. $\square$

5. (*5 pts*) Given a matrix $A \in \mathbb{R}^{n\times n}$, define what it means for A to be *positive definite.*

$x^TAx \geq 0$ for all $x \in \mathbb{R}^n$

and

$x^TAx > 0$ for $x \neq 0$

6. **(a)** (*4 pts*) Define *convex set* (for a subset S of $\mathbb{R}^n$).

if $x, y \in S$ and $0 \leq \alpha \leq 1$ then

$\alpha x + (1-\alpha) y \in S$.

(b) (*4 pts*) Define *convex function* (for a scalar valued function $f(x)$).

$f: S \to \mathbb{R}$ is convex if S is convex

and if $x, y \in S$ and $0 \leq \alpha \leq 1$

implies $f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$.

7. (*5 pts*) For a linear programming problem in standard form, define *basic feasible solution.*

for $\min c^Tx$ s.t. $Ax=b$, $x \geq 0$, a vector x

is a basic feasible solution if $Ax=b$, $x\geq 0$,

and columns of A corresponding to $x_i > 0$ are linearly - independent

8. **(a)** (*6 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = 3x_1 - 9x_2 \\ \text{subject to} & 5x_1 + 2x_2 \leq 30 \\ & 3x_1 - x_2 \geq -4 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

$5x + 2y \leq 30$

$y \leq 15 - \frac{5}{2}x$

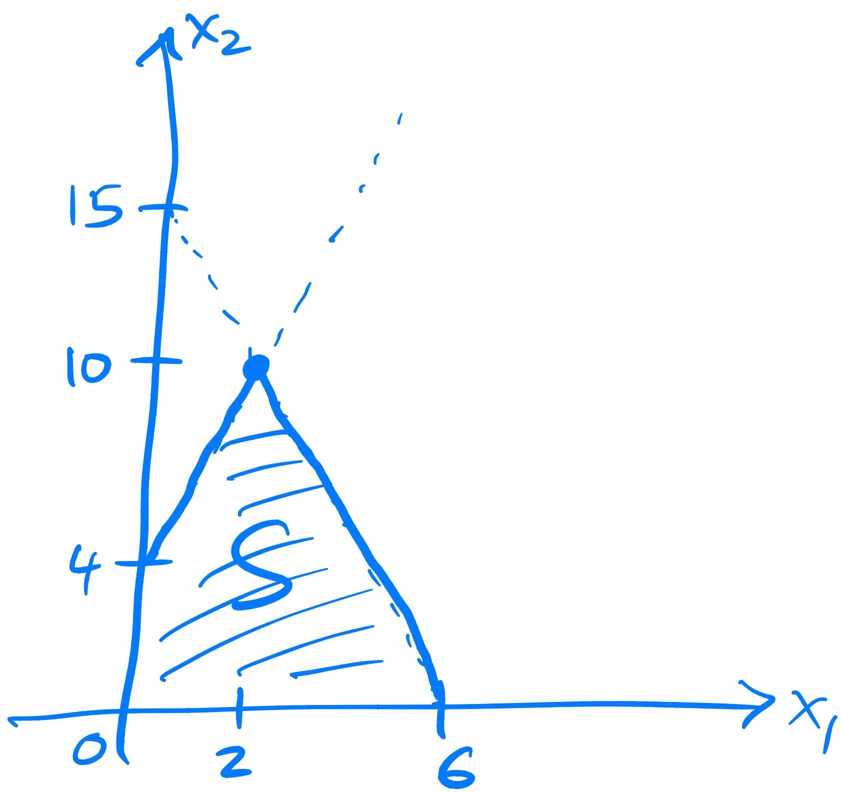
$-3x + y \leq 4$

$y \leq 4 + 3x$

$15 - \frac{5}{2}x = 4 + 3x$

$11 = (3 + \frac{5}{2})x = \frac{11}{2}x$

$x = 2 \Rightarrow y = 10$



(b) (*6 pts*) Convert the problem in **(a)** to standard form.

$$\begin{array}{ll} \text{min} & 3x_1 - 9x_2 + 0x_3 + 0x_4 \\ \text{s.t.} & 5x_1 + 2x_2 + x_3 = 30 \\ & -3x_1 + x_2 + x_4 = 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{array}$$

So

$$A = \begin{bmatrix} 5 & 2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 30 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -9 \\ 0 \\ 0 \end{bmatrix}$$

(c) (*8 pts*) Let x be the basic feasible solution to the standard-form problem, as computed in **8(b)**, for which $x_1 = 0$ and $x_2 = 0$. Use the template to complete one iteration of the (reduced) simplex method. **At the bottom**, fill in the basic and non-basic variables (indices) at the completion of this first iteration.

$$\mathcal{B} = \{\, 3, 4 \,\}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underline{B x_B = b} \implies x_B = \hat{b} = \begin{bmatrix} 30 \\ 4 \end{bmatrix}$$

$$\mathcal{N} = \{\, 1, 2 \,\}, \quad N = \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}, \quad c_N = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -9 \end{bmatrix} - N^\top \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \implies \qquad \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

$\cancel{\boxed{\hat{c}_N \geq 0\text{?: stop with optimum}}}$ $\quad \hat{c}_N \xrightarrow{\text{index of}}_{\text{min}} \quad t = \boxed{2} \rightarrow \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\boxed{\hat{A}_t \leq 0\text{?: stop, unbounded}}$ $\quad \left\{ \frac{\hat{b}_i}{\hat{a}_{i,t}} \right\} = \left\{ \frac{30}{2}, \frac{4}{1} \right\} \quad \xrightarrow{\text{index of}}_{\text{min over } \hat{a}_{i,t} > 0} \quad s = \boxed{4}$

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result: $\quad \mathcal{B} = \{\, 3, 2 \,\}, \quad \mathcal{N} = \{\, 1, 4 \,\}$

9. (*5 pts*) Given a linear programming problem in standard form

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

What is the dual problem?

maximize $b^T y$

s.t. $A^T y \leq c$

10. (*10 pts*) Prove: **Theorem.** *Let x_* be a local minimizer of a convex optimization problem. Then x_* is also a global minimizer.*

Proof. Suppose x_* is not a global minimizer, so there is a feasible $y \in S$, with $y \neq x_*$, so that $f(y) < f(x_*)$.

Let $0 < \alpha < 1$. Since S is convex,

$x_* + \alpha(y - x_*) = (1-\alpha)x_* + \alpha y \in S$ is also feasible. But since f is convex

$$\begin{aligned} f((1-\alpha)x_* + \alpha y) &\leq (1-\alpha) f(x_*) + \alpha f(y) \\ &< (1-\alpha) f(x_*) + \alpha f(x_*) \\ &= f(x_*). \end{aligned}$$

We can choose $\alpha > 0$ so that $x_* + \alpha(y - x_*) = z$ is as close to x_* as desired, but $f(z) < f(x_*)$. Thus x_* is not a local minimizer. By contradiction, x_* is a global minimizer. $\square$