Name: ____________________________________

Math 661 Optimization (Bueler) Friday, 28 October 2022

Midterm Exam

In class. No book. No calculator. 1/2 **sheet of notes allowed.**

(100 *points possible*)

1. Consider Newton's method to solve the scalar equation $f(x) = 0$.

(a) (*8 pts*) Draw and label a sketch of one step of Newton's method. In particular, your graph should show $y = f(x)$ as a generic curve, then an iterate x_k, and then show (graphically) how the next iterate x_{k+1} is determined.

(b) (*5 pts*) Do one step of Newton's method to solve the equation $x^3 - x + 1 = 0$, starting at $x_0 = 1$. That is, compute x_1.

2. Let $f(x) = 4x_3x_2 + x_3^2 + x_2 - 2x_1^2$ for $x \in \mathbb{R}^3$.

(a) (*4 pts*) Compute the gradient and Hessian of f at $x_k = (-1, 1, 1)^\top \in \mathbb{R}^3$.

(b) (*4 pts*) Does $f(x)$ have any stationary points? If so, find them.

(c) (*4 pts*) Find all the local minima x_* of f, or explain why none exist. Justify your answer using appropriate 1st- or 2nd-order necessary or sufficient conditions.

(d) (*4 pts*) Is $p = (-2, 1, 0)^\top$ a descent direction for f at x_k from part **(a)**?

3. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \exp(x_1^4 + x_2^2) - x_1^4 + \sin(x_1 x_2 x_3) \\ \text{subject to} & 2x_1 - 2x_2 + x_3 = -1 \\ & x_1 + 4x_2 \geq -3 \\ & 7x_2 - 5x_3 \geq -1 \end{array}$$

(a) (*4 pts*) Is $x = (-2, 0, 3)^\top$ feasible?

(b) (*4 pts*) Considering both equality and inequality constraints, which constraints are active and which are inactive at $x = (1, 3, 3)^\top$?

Extra Credit. (*3 pts*) For $x \in \mathbb{R}^n$, completely solve the standard-form linear programming problem when there are no equality constraints:

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & x \geq 0 \end{array}$$

4. (*10 pts*) Consider general minimization problems of the form

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & a_j^\top x = b_j \quad \text{for } j \in \mathcal{E} \\ & a_j^\top x \geq b_j \quad \text{for } j \in \mathcal{I} \end{aligned}$$

for given vectors $a_j \in \mathbb{R}^n$ and scalars b_j.

Suppose $\bar{x}$ is a point in the feasible set. Let $\hat{\mathcal{I}}$ be the set of indices $j \in \mathcal{I}$ where the inequality constraint $a_j^\top x \geq b_j$ is active at $\bar{x}$. Show that if $a_j^\top p = 0$ for all $j \in \mathcal{E}$, and if $a_j^\top p \geq 0$ for all $j \in \hat{\mathcal{I}}$, then p is a feasible direction.

5. (*5 pts*) Given a matrix $A \in \mathbb{R}^{n\times n}$, define what it means for A to be *positive definite.*

6. **(a)** (*4 pts*) Define *convex set* (for a subset S of $\mathbb{R}^n$).

(b) (*4 pts*) Define *convex function* (for a scalar valued function $f(x)$).

7. (*5 pts*) For a linear programming problem in standard form, define *basic feasible solution.*

8. **(a)** (*6 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = 3x_1 - 9x_2 \\ \text{subject to} & 5x_1 + 2x_2 \leq 30 \\ & 3x_1 - x_2 \geq -4 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(b) (*6 pts*) Convert the problem in **(a)** to standard form.

(c) (*8 pts*) Let x be the basic feasible solution to the standard-form problem, as computed in **8(b)**, for which $x_1 = 0$ and $x_2 = 0$. Use the template to complete one iteration of the (reduced) simplex method. **At the bottom**, fill in the basic and non-basic variables (indices) at the completion of this first iteration.

$$\mathcal{B} = \{ \qquad \}, \quad B = \begin{bmatrix} \qquad \end{bmatrix}, \quad c_B = \begin{bmatrix} \quad \end{bmatrix}, \quad \underline{Bx_B = b} \implies x_B = \hat{b} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{N} = \{ \qquad \}, \quad N = \begin{bmatrix} \qquad \end{bmatrix}, \quad c_N = \begin{bmatrix} \quad \end{bmatrix}$$

$$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} \quad \end{bmatrix} \implies \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} \quad \end{bmatrix}$$

| $\hat{c}_N \geq 0$?: stop with optimum |

$$\hat{c}_N \xrightarrow[\text{min}]{\text{index of}} t = \boxed{} \rightarrow \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

| $\hat{A}_t \leq 0$?: stop, unbounded |

$$\left\{ \frac{\hat{b}_i}{\hat{a}_{i,t}} \right\} = \{ \qquad \} \xrightarrow[\text{min over } \hat{a}_{i,t} > 0]{\text{index of}} s = \boxed{}$$

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result: $\mathcal{B} = \{ \qquad \}, \quad \mathcal{N} = \{ \qquad \}$

9. (*5 pts*) Given a linear programming problem in standard form

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

What is the dual problem?

10. (*10 pts*) Prove: **Theorem.** *Let x_* be a local minimizer of a convex optimization problem. Then x_* is also a global minimizer.*

Proof.