Name: ______________________________

Math 661 Optimization (Bueler) Friday, 26 October 2018

Midterm Exam

In class. No book. No calculator. 1/2 sheet of notes allowed.

(100 *points possible*)

1. Let $f(x) = x_1^3 + x_1^2 x_2 + x_3$ for $x \in \mathbb{R}^3$.

(a) (*10 pts*) Compute the gradient and Hessian of f at $x_k = (-1, 1, 1)^\top \in \mathbb{R}^3$.

(b) (*5 pts*) Show that f has no local minima. (*Hint: Explain using an appropriate 1st- or 2nd-order necessary or sufficient condition.*)

(c) (*5 pts*) Is $p = (1, 2, 3)^\top$ a descent direction for f at x_k from part **(a)**?

2. **(a)** (*5 pts*) Does the sequence of real numbers defined by $x_0 = 1$ and

$$x_{k+1} = \frac{1}{k+1} x_k,$$

for $k = 0, 1, 2, \ldots$, converge to $x_* = 0$ superlinearly? Show why your answer is true.

(b) (*5 pts*) Show that the sequence

$$x_k = 3^{(-2^k)}$$

converges quadratically. (*Hint: Start by identifying the point x_* to which the sequence converges.*)

3. (*10 pts*) Consider $f(x) = \frac{1}{4}x^4 - x^2 + 2x$ and $x_0 = 1$. Compute x_1 from Newton's method for the optimization problem $\min_{x \in \mathbb{R}} f(x)$.

Extra Credit. (*3 pts*) Consider the equation $f(x) = 0$ where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is twice continuously differentiable. Assume that the iterates x_k from Newton's method converge to x_* satisfying the equation. Also assume that $f'(x_*) \neq 0$. Show that $x_k \to x_*$ quadratically. (*Use space on the back page, or a separate sheet of paper.*)

4. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \cos(x_1) - x_2^3 + \exp(x_1^4 + x_2^2) \\ \text{subject to} & 2x_1 - 4x_2 + x_3 = -1 \\ & x_1 + 4x_2 \geq -3 \\ & 7x_2 - 5x_3 \geq 2 \end{array}$$

(a) (*3 pts*) Is $x = (1, 1, 1)^\top$ feasible?

(b) (*3 pts*) Which constraints are active and which are inactive at $x = (1, 1, 1)^\top$?

(c) (*10 pts*) Consider general problems of the same form, with linear equality constraints $a_j^\top x = b_j$ for $j \in \mathcal{E}$ and inequality constraints $a_i^\top x \geq b_i$ for $i \in \mathcal{I}$, and assume $\bar{x}$ is feasible. Show that if $a_j^\top p = 0$ for all $j \in \mathcal{E}$ and $a_i^\top p \geq 0$ for all $i \in \hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the set of indices where the constraint $a_i^\top x \geq b_i$ is active at $\bar{x}$, then p is a feasible direction.

(d) (*4 pts*) *Fill in the blanks*: For the same general class of problems as described in **(c)**, and assuming x is feasible and that p is a feasible direction, the __________________ determines the maximum value of α so that $x + \alpha p$ is feasible. Only the inequality constraints which are _______________ are needed for this test.

5. **(a)** (*8 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = -2x_1 + x_2 \\ \text{subject to} & -2x_1 + 3x_2 \leq 3 \\ & -x_1 + x_2 \geq -1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(b) (*7 pts*) Convert the problem in **(a)** to standard form.

(c) (*10 pts*) Let x be the basic feasible solution, to the problem in **(b)**, for which $x_1 = 0$ and $x_2 = 0$. Use the template to complete one iteration of the (reduced) simplex method. In particular, at bottom, fill in the basic and non-basic variables (indices) at the *completion* of this first iteration.

$\mathcal{B} = \{ \quad \}$, $B = \begin{bmatrix} \quad \end{bmatrix}$, $c_B = \begin{bmatrix} \quad \end{bmatrix}$, $\underline{Bx_B = b} \implies x_B = \hat{b} = \begin{bmatrix} \quad \end{bmatrix}$

$\mathcal{N} = \{ \quad \}$, $N = \begin{bmatrix} \quad \end{bmatrix}$, $c_N = \begin{bmatrix} \quad \end{bmatrix}$

$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} \quad \end{bmatrix} \implies \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} \quad \end{bmatrix}$

$\boxed{\hat{c}_N \geq 0\text{?: stop with optimum}}$ $\quad \hat{c}_N \xrightarrow[\text{min}]{\text{index of}} t = \boxed{} \rightarrow \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

$\boxed{\hat{A}_t \leq 0\text{?: stop, unbounded}}$ $\quad \left\{ \frac{\hat{b}_i}{\hat{a}_{i,t}} \right\} = \{ \quad \} \xrightarrow[\text{min over } \hat{a}_{i,t} > 0]{\text{index of}} s = \boxed{}$

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result: $\quad \mathcal{B} = \{ \quad \}$, $\quad \mathcal{N} = \{ \quad \}$

6. (a) (*5 pts*) Given a linear programming problem in standard form

$$\begin{aligned} \text{minimize} \quad & z = c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

What is the dual problem?

(b) (*5 pts*) Fill in the blank to state the following theorem *and* prove the theorem. Please justify any inequalities.

Theorem (weak duality). *Let x be a feasible point for the primal problem in standard form, and let y be a feasible point for the dual problem. Then*

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Proof.

7. (*5 pts*) Given a matrix $A \in \mathbb{R}^{n\times n}$, define what it means for A to be *positive definite.*

BLANK PAGE FOR SCRATCH WORK. CLEARLY-LABEL ANYTHING YOU WANT TO BE GRADED