Name: SOLUTIONS

Math 661 Optimization (Bueler)

Friday, 26 October 2018

Midterm Exam

In class. No book. No calculator. 1/2 **sheet of notes allowed.**

(100 *points possible*)

1. Let $f(x) = x_1^3 + x_1^2 x_2 + x_3$ for $x \in \mathbb{R}^3$.

(a) (*10 pts*) Compute the gradient and Hessian of f at $x_k = (-1, 1, 1)^\top \in \mathbb{R}^3$.

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 + 2x_1 x_2 \\ x_1^2 \\ 1 \end{bmatrix} \longrightarrow \nabla f(x_k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 6x_1 + 2x_2 & 2x_1 & 0 \\ 2x_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \nabla^2 f(x_k) = \begin{bmatrix} -4 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) (*5 pts*) Show that f has no local minima. (*Hint: Explain using an appropriate 1st- or 2nd-order necessary or sufficient condition.*)

f has no local min. because $\nabla f(x) \neq 0$ for any x.

(this uses the 1st-order necessary condition)

note 3rd entry of $\nabla f(x)$ is 1 always

(c) (*5 pts*) Is $p = (1, 2, 3)^\top$ a descent direction for f at x_k from part **(a)**?

we compute

$$\nabla f(x_k)^T p = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 6 > 0$$

but "$\nabla f(x)^T p < 0$" for descent directions,

so no

2. **(a)** (*5 pts*) Does the sequence of real numbers defined by $x_0 = 1$ and

$$x_{k+1} = \frac{1}{k+1} x_k,$$

for $k = 0, 1, 2, \ldots,$ converge to $x_* = 0$ superlinearly? Show why your answer is true.

$e_K = X_K - X_* = X_K \leftarrow$ (in this case)

$$\lim_{K\to\infty} \frac{\|e_{K+1}\|}{\|e_K\|} = \lim_{K\to\infty} \frac{\left|\frac{1}{K+1}\cancel{X_K}\right|}{|\cancel{X_K}|} = \lim_{K\to\infty} \frac{1}{K+1} = 0$$

so $\{X_K\}$ converges superlinearly

(b) (*5 pts*) Show that the sequence

$$x_k = 3^{(-2^k)}$$

converges quadratically. (*Hint: Start by identifying the point x_* to which the sequence converges.*)

$$X_* = \lim_{K\to\infty} X_K = 0 \qquad \left(= \lim_{n\to\infty} 3^{-n}\right)$$

So $$\lim_{K\to\infty} \frac{\|e_{K+1}\|}{\|e_K\|^2} = \lim_{K\to\infty} \frac{3^{(-2^{K+1})}}{3^{(-2^K)2}} = \lim_{K\to\infty} 1 = 1 = c$$

and $0 < c < \infty$ so $\{X_K\}$ converges quadratically

3. (*10 pts*) Consider $f(x) = \frac{1}{4}x^4 - x^2 + 2x$ and $x_0 = 1$. Compute x_1 from Newton's method for the optimization problem $\min_{x\in\mathbb{R}} f(x)$.

$$X_1 = X_0 - \frac{f'(X_0)}{f''(X_0)} = 1 - \frac{1 - 2 + 2}{3 - 2} = 1 - \frac{1}{1} = 0$$

$f'(x) = x^3 - 2x + 2$

$f''(x) = 3x^2 - 2$

Extra Credit. (*3 pts*) Consider the equation $f(x) = 0$ where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is twice continuously differentiable. Assume that the iterates x_k from Newton's method converge to x_* satisfying the equation. Also assume that $f'(x_*) \neq 0$. Show that $x_k \to x_*$ quadratically. (*Use space on the back page, or a separate sheet of paper.*)

see back

4. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \cos(x_1) - x_2^3 + \exp(x_1^4 + x_2^2) \\ \text{subject to} & 2x_1 - 4x_2 + x_3 = -1 \\ & x_1 + 4x_2 \geq -3 \\ & 7x_2 - 5x_3 \geq 2 \end{array}$$

(a) (*3 pts*) Is $x = (1,1,1)^\top$ feasible?

$2 \cdot 1 - 4 \cdot 1 + 1 = -1$ ✓

$5 = 1 + 4 \cdot 1 \geq -3$ ✓

$2 = 7 - 5 \geq 2$ ✓

yes

(b) (*3 pts*) Which constraints are active and which are inactive at $x = (1,1,1)^\top$?

equality constraint is active

1st inequality constraint is inactive

2nd inequality constraint is active

(c) (*10 pts*) Consider general problems of the same form, with linear equality constraints $a_j^\top x = b_j$ for $j \in \mathcal{E}$ and inequality constraints $a_i^\top x \geq b_i$ for $i \in \mathcal{I}$, and assume $\bar{x}$ is feasible. Show that if $a_j^\top p = 0$ for all $j \in \mathcal{E}$ and $a_i^\top p \geq 0$ for all $i \in \hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the set of indices where the constraint $a_i^\top x \geq b_i$ is active at $\bar{x}$, then p is a feasible direction.

$j \in \mathcal{E}$:

$$a_j^T(\bar{x} + \alpha p) = a_j^T \bar{x} + \alpha a_j^T p = b_j + 0 = b_j$$

$i \in \mathcal{I}$:

$$a_i^T(\bar{x} + \alpha p) = a_i^T \bar{x} + \alpha a_i^T p \geq b_i + 0 = b_i$$

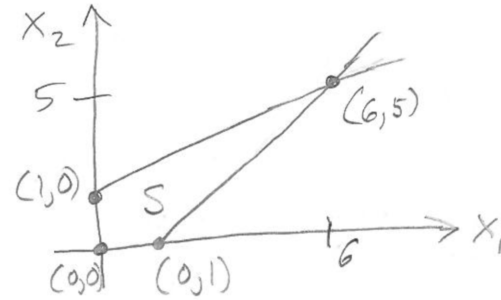
so $\bar{x} + \alpha p$ is feasible so p is a feasible direction

(d) (*4 pts*) *Fill in the blanks*: For the same general class of problems as described in **(c)**, and assuming x is feasible and that p is a feasible direction, the ratio test determines the maximum value of α so that $x + \alpha p$ is feasible. Only the inequality constraints which are inactive are needed for this test.

5. **(a)** (*8 pts*) Sketch the feasible set for the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & z = -2x_1 + x_2 \\ \text{subject to} & -2x_1 + 3x_2 \le 3 \\ & -x_1 + x_2 \ge -1 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

$-2x+3y=3$
$y = 1+\frac{2}{3}x$
$-x+y=-1$
$y=-1+x$



$1+\frac{2}{3}x = -1+x$
$2 = \frac{1}{3}x$
$x = 6$

(b) (*7 pts*) Convert the problem in **(a)** to standard form.

$$\begin{array}{lllll} \text{min.} & z = -2x_1 + x_2 + 0x_3 + 0x_4 & & & \\ \text{s.t.} & -2x_1 + 3x_2 + x_3 & & = 3 & \\ & x_1 - x_2 & + x_4 & = 1 & \\ & x_1, x_2, x_3, x_4 \ge 0 & & & \end{array}$$

$A = \begin{bmatrix} -2 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$c = [-2 \ 1 \ 0 \ 0]^T$

(c) (*10 pts*) Let x be the basic feasible solution, to the problem in **(b)**, for which $x_1 = 0$ and $x_2 = 0$. Use the template to complete one iteration of the (reduced) simplex method. In particular, at bottom, fill in the basic and non-basic variables (indices) at the *completion* of this first iteration.

$\mathcal{B} = \{3, 4\}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $c_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\underline{Bx_B = b} \implies x_B = \hat{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\mathcal{N} = \{1, 2\}$, $N = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$, $c_N = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\underline{B^\top y = c_B} \implies y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \underline{\hat{c}_N = c_N - N^\top y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\boxed{\hat{c}_N \ge 0?\text{: stop with optimum}}$ $\quad \hat{c}_N \xrightarrow[\text{min}]{\text{index of}} t = \boxed{1} \to \underline{B\hat{A}_t = A_t} \implies \hat{A}_t = \begin{bmatrix} \hat{a}_{1,t} \\ \vdots \\ \hat{a}_{m,t} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

A_1

$\boxed{\hat{A}_t \le 0?\text{: stop, unbounded}}$ $\quad \left\{\frac{\hat{b}_i}{\hat{a}_{i,t}}\right\} = \left\{ \cancel{\frac{3}{-2}}, \frac{1}{1} \right\} \xrightarrow[\text{min over } \hat{a}_{i,t} > 0]{\text{index of}} s = \boxed{4}$

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result: $\mathcal{B} = \{3, 1\}$, $\mathcal{N} = \{4, 2\}$

6. **(a)** (*5 pts*) Given a linear programming problem in standard form

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

What is the dual problem?

$$\begin{array}{ll} \text{maximize} & w = b^T y \\ \text{s. t.} & A^T y \leq c \end{array}$$

(b) (*5 pts*) Fill in the blank to state the following theorem *and* prove the theorem. Please justify any inequalities.

Theorem (weak duality). *Let x be a feasible point for the primal problem in standard form, and let y be a feasible point for the dual problem. Then* $\underline{z = c^T x \geq b^T y = w}$

Proof. because $x \geq 0$,

$$z = c^T x \geq (A^T y)^T x = y^T (Ax) = y^T b = w$$

$\boxed{/}$

7. (*5 pts*) Given a matrix $A \in \mathbb{R}^{n \times n}$, define what it means for A to be *positive definite*.

<u>def.</u> $A \in \mathbb{R}^{n \times n}$ is <u>positive definite</u> if

$p^T A p > 0$ for all non zero vectors p

BLANK PAGE FOR SCRATCH WORK. CLEARLY-LABEL ANYTHING YOU WANT TO BE GRADED

Extra Credit We know $f(x_*)=0$ and $e_k = x_k - x_*$ has limit zero. By Taylor's theorem,

$$0 = f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\xi)(x_* - x_k)^2$$

Equivalently

$$-f(x_k) + f'(x_k)x_k - f'(x_k)x_* = \frac{1}{2} f''(\xi) e_k^2.$$

or, assuming $f'(x_k) \neq 0$,

$$\underbrace{\left(x_k - \frac{f(x_k)}{f'(x_k)}\right)}_{= x_{k+1} \text{ by Newton's method.}} - x_* = \frac{f''(\xi)}{2f'(x_k)} e_k^2$$

Thus

$$e_{k+1} = \frac{f''(\xi)}{2f'(x_k)} e_k^2.$$

As $k \to \infty$ we know $x_k \to x_*$ so $\xi \to x_*$. Since $f'(x)$ is continuous and $f'(x_*) \neq 0$, $f'(x_k) \neq 0$ for large k. Thus

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = \lim_{k\to\infty} \frac{f''(\xi)}{2f'(x_k)} = \frac{f''(x_*)}{2f'(x_*)} = C.$$

We know $C < \infty$. Thus $x_k \to x_*$ (at least) quadratically.