Final Exam

Due 5pm on Thursday 12/13/2018, in my Chapman 101 office box.

100 points total.

As stated on the syllabus, this exam is 25% of your course grade.

Rules. This take-home exam must be *your own work*. You may not talk or communicate about it with any person other than me, Ed Bueler. However, you *are* encouraged to ask me questions about the exam during lecture time, and also during my office hours. You may use any reference book or article, print or electronic, as long as it is clearly cited. Specific references to the textbook[1] are strongly recommended.

Pre-existing codes. You *may* use codes posted at the class webpage,

`bueler.github.io/M661F18/`

Concretely, problems on this Final Exam ask you to use codes `mysimplex.m` and `sdbt.m`, or codes which you have written for the same purpose. If you use one of your own codes then please clearly identify or show it, and show how you call/use it.

F1. Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & z = -2x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 + x_2 \geq -3 \\ & -2x_1 + x_2 \leq 1 \\ & x_1 - 2x_2 \geq -5 \\ & x_1 + x_2 \leq 7 \\ & x \geq 0 \end{aligned}$$

(a) (*5 pts*) Sketch the feasible set for the above problem. Be sure to label the axes, label the feasible set S, and give coordinates for each extreme point of S.

(b) (*5 pts*) Put the above problem in standard form.

(c) (*5 pts*) Apply the simplex method to the above problem. Note that a basic feasible solution can be found from the standard form of the problem. You *may* use the code `mysimplex.m`, which is posted online, or a version of the same algorithm which you have previously written, but please identify the code you are using and show how you use it. Finally, use the resulting solution to label the sequence of iterates from the simplex method on your sketch in **(a)**.

F2. Suppose $c \in \mathbb{R}^n$ is a nonzero vector and consider the problem

$$\begin{aligned} \text{minimize} \quad & z = c^\top x \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i^2 = 1 \end{aligned}$$

where $x \in \mathbb{R}^n$. Note there is a single equality constraint, which can be written $\|x\|^2 = 1$.

(a) (*5 pts*) By arguing informally explain why the solution is

$$x_* = -\frac{c}{\|c\|}.$$

[1] Griva, Nash & Sofer, *Linear and Nonlinear Optimization*, 2nd ed. 2009

Use a sketch of the $n = 2$ case to explain.

(b) (*5 pts*) The necessary optimality conditions for this problem are addressed by Theorem 14.15 on page 504 of the textbook. Compute the Lagrangian and state the first-order necessary conditions in detail. (*Note that you do* not *need to compute the null-space matrix $Z(x_*)$ in order to do this.*)

(c) (*5 pts*) Solve the conditions in **(b)** algebraically to confirm the solution in part **(a)**. How many points (x_*, λ_*) are there which satisfy the first-order necessary conditions?

F3. (*Before doing this problem read Example 14.20 on pages 506–507. This problem asks for a similar analysis.*) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \le 4 \\ & x_2 \ge 0 \end{aligned}$$

(a) (*5 pts*) Sketch the feasible set and explain informally, perhaps using contours of f, why $x_* = (1, 0)^\top$ is the solution.

(b) (*10 pts*) Write the constraints in the form $g_i(x) \ge 0$. Compute the Lagrangian and its gradient. For each of the points $A = (0, 0)^\top$, $B = (0, 2)^\top$, and $C = (1, 0)^\top$ compute the values of λ_i satisfying the zero-gradient condition. Address whether these points satisfy the first-order optimality conditions, that is, whether they are candidates for a local minimizer. Show in particular that C satisfies all the first-order conditions in Theorem 14.18. (*Note that you do* not *need to find any null-space matrices $Z(x)$ in order to answer this question.*)

F4. Consider the 2D unconstrained minimization problem

$$\text{minimize} \quad f(x) = 3\sin(x_1) + \cos(x_2) + \frac{1}{20}\left(x_1^2 - x_1 x_2 + 2x_2^2\right).$$

This example is intended to be representative of smooth optimization problems where there are many local minima. For functions like this the algorithms we have studied in Math 661 generally only find a local minimum which is nearby to the initial iterate x_0.

(a) (*5 pts*) Consider the set $R = \{-10 < x_1 \le 10, -10 < x_2 \le 10\}$. Use MATLAB (or etc.) to generate a surface plot of f on R. This plot includes the global minimum and several other local minima. By examining this plot, give approximate coordinates of the global minimum x_*.

(b) (*10 pts*) Suppose we only know that the minimum occurs somewhere in R. Write a "grid search" program which first generates 400 distinct initial iterates x_0 which are exactly the points in R with integer entries. The code then exhaustively searches from these initial iterates. Specifically, for each x_0 the program calls the steepest-descent code `sdbt.m`.[2] Thereby find the global minimum x_* to at least 6 digits. Identify which starting point(s) x_0 led to x_*. Display the grid search code you have written.

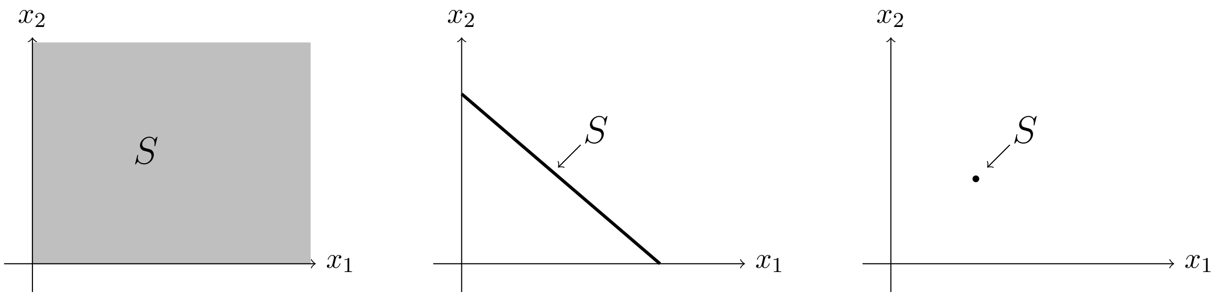
F5. (*15 pts*) Consider nonlinear optimization problems on $x \in \mathbb{R}^n$ which have standard-form linear constraints:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

Assume f is smooth and that there are m scalar constraint equations, i.e. $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

[2] Or a version of the same algorithm which you have previously written. In that case please identify the code you are using and show how you use it.

In 2D ($n = 2$) there are three possibilities for the dimension of the feasible set. The cartoons below illustrate these three possibilities in the cases where the feasible sets S are non-empty, generic, and bounded. ("Bounded" applies for $m > 0$.)



$m = 0$ $m = 1$ $m = 2$

For 3D ($n = 3$) there are four possibilities $m = 0, 1, 2, 3$. Sketch the four corresponding cartoons. These cartoons should have the same annotations as the 2D versions above.

F6. (*This problem asks you to write the general KKT conditions.*) Consider the general nonlinear constrained optimization problem

$$\begin{array}{rll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) = 0, & i = 1, \dots, \ell \\ & h_i(x) \geq 0, & i = 1, \dots, m \end{array}$$

(a) (*5 pts*) In section 14.5 the meaning of the phrase "*x_* is a regular point of the constraints*" is given. State this definition precisely for the above problem. Also state the Lagrangian for this problem.

(b) (*5 pts*) Suppose x_* is a local minimizer of the above problem which is also a regular point. State the first-order necessary conditions for the above problem. (*Again note that you do* not *need to find any null-space matrices $Z(x)$ in order to answer this question.*)

F7. (*15 pts*) Exercise #2.7 in section 14.2. (*Use techniques from either section 14.2 or 14.3.*)