IEEE 754: What it means for humanity and your computer

Our textbook sketches how actual floating point systems work. Here I lay out the basic details of how floating point numbers are *actually* implemented on a computer.

- Computer memories are organized into *bits* and *bytes*. A bit is a binary digit, the irreducible atom of memory; it is always in either of two states $\{0, 1\}$. A byte is a group of 8 bits.

- *Integers* are represented on computers using 1, 2, 4, or 8 bytes. How this is done is straightforward, but we ignore the details.

- The IEEE 754 standard is about how *real* numbers are approximately represented in memory, that is, how *floating point* numbers are represented. "Floating point" is essentially just scientific notation, but using bits. Since computer memories are finite, only a finite subset of real numbers can be represented.

- "IEEE" stands for "Institute of Electrical and Electronics Engineers". For more information on the standard than described here, see the wikipedia page
  `en.wikipedia.org/wiki/IEEE_754`

- The most important floating point representations use 32 and 64 bits, or 4 and 8 bytes, respectively. These are called binary32 ("`single`") and binary64 ("`double`") in the standard, respectively. In binary32 the number

$$x = (-1)^s \times (1.d_1 d_2 d_3 \dots d_{23})_2 \times 2^{(e_1 \dots e_8)_2 - 127}$$

is represented by 32 bits this way:

s	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}

In binary64, a.k.a. `double`, the number

$$x = (-1)^s \times (1.d_1 d_2 d_3 \dots d_{52})_2 \times 2^{(e_1 \dots e_{11})_2 - 1023}$$

is represented by 64 bits this way:

s	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	$\cdots$	d_{51}	d_{52}

- Note that the "1." in the above representations, which appears before the d_i bits, is always present and therefore it does *not* use a bit of memory! It is called the "implicit leading bit".

- The IEEE 754 standard uses abstract language to describe the way the bits are arranged. Concretely, every representable *nonzero* number is of the form

(1)
$$x = (-1)^s \times \frac{m}{2^{t-1}} \times 2^e$$

for fixed positive integer t (the *precision*). The other symbols, namely $s \in \{0, 1\}$ (the *sign*), the integer m (the *mantissa*), and the integer e (the *exponent*), determine, x. These satisfy

(2)
$$2^{t-1} \le m \le 2^t - 1, \qquad e_{min} \le e \le e_{max}.$$

- In the current version of the standard, IEEE 754-2008,[1] there are a number of formats. However, we ignore the *decimal* standards, which are rarely used, and look only at *binary* formats. The four formats that matter most use 16, 32, 64, or 128 bits, respectively. We have already shown how 32 and 64 bit floating point numbers are implemented in memory. In terms of form (1) and constraints (2), they follow this table:

name	common name	precision t	exponent bits	exponent bias	e_{min}	e_{max}
binary16	`half`	11	5	$2^4 - 1 = 15$	-14	+15
binary32	`single`	24	8	$2^7 - 1 = 127$	-126	+127
binary64	`double`	53	11	$2^{10} - 1 = 1023$	-1022	+1023
binary128	`quadruple`	113	15	$2^{14} - 1 = 16383$	-16382	+16383

- In decimal (base 10) you get these values:

name	decimal precision	decimal e_{max}	decimal e_{min}
binary16	3.31	4.51	-4.21
binary32	7.22	38.23	-37.93
binary64	15.95	307.95	-307.65
binary128	34.02	4931.77	-4931.47

- Regarding the exponent, if all bits e_i are zero or all are one then the number has special meaning. Otherwise, for *normal* numbers, in `single` the standard requires $(e_1 \dots e_8)_2 \in \{1, 2, \dots, 254\}$ and in `double` the standard requires $(e_1 \dots e_{11})_2 \in \{1, 2, \dots, 2046\}$.

- Representing the number zero, which is *not* in form (1), is an example of "special meaning." It is done by setting all bits other than s to zero. Because the sign bit is not determined, this means "$+0$" and "-0" exist as separate representations. (Strange but true!)

- There are representation standards for weird stuff:
  - $+\infty$ and $-\infty$
  - things that are "not a number" (`NaN`)
  - things called "subnormal" numbers

  For a subnormal number in the `single` representation, for example, the exponent bits are all zero, but some bits d_i are nonzero.

- The IEEE 754 standard also addresses the rounding errors which occur in arithmetic operations (addition, multiplication, etc.). The details are not in the scope of this note.

- It is safe to assume your laptop implements IEEE-compliant binary64 floating point operations *in hardware*. Other types are commonly implemented in software, especially binary128, which is thus much slower on current computers. The smaller binary16 and binary32 formats are typically used for *storing* numbers, which increases storage capacity.

- In the last 10 years, 16 bit standards have been resurgent because machine learning can be faster with lower precision, and it does not affect much about the result. See the "bfloat16 floating-point format" wikipedia page, for example.

[1] IEEE 754-2019 is a minor revision.