Name: ______________________________

Math 310 Numerical Analysis (Bueler) December 2019

SAMPLE Final Exam

In class. No book or electronics. 1/2 sheet of notes allowed. 120 minutes maximum.

1. Write a MATLAB code for the Newton method applied to the problem $f(x) = 0$:

```
function x = newton(f,dfdx,x0)
```

The inputs are $f =$ `f`, the derivative $f' =$ `dfdx`, and an initial estimate $x_0 =$ `x0`. Stop the algorithm (*show this in the code!*) when $|f(x)| \le 10^{-6}$.

2. **(a)** State the polynomial interpolation error theorem (with a remainder term). *Carefully state the hypotheses* and *the conclusion of the theorem.*

(b) Assume the interval in question is $[-1, 1]$ and that the interpolation points are the Chebyshev points $x_j = \cos(\pi j/n)$ for $n = 0, 1, 2, \dots, n$. What can you say about the remainder term that explains why the Chebyshev points are effective for interpolation? *Answer in a couple of complete sentences.*

3. (a) Consider

$$f(x) = \frac{1}{x+2}.$$

Completely set up, but do not solve, the Vandermonde system to find the degree 3 polynomial $p(x)$ which interpolates $f(x)$ at the points $x_0 = -1.5, x_1 = -1, x_2 = 0, x_3 = 1$.

(b) For the same $f(x)$ and interpolation points as in part **(a)**, write down Lagrange's form of the polynomial $p(x)$. Do not simplify.

4. Table 10.3 includes the error formula for Simpson's rule: if $f \in C^4[a,b]$ then

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{1}{2880}(b-a)^5 f^{(4)}(\xi)$$

for some $\xi \in [a,b]$. Why does this fact show that Simpson's rule is exact if $f(x)$ is a cubic polynomial? *Answer in at least one complete sentence.*

5. (a) Find A_0 and A_1 so that the numerical integration rule

$$\int_{-1}^{1} f(x)\,dx \approx A_0 f(-\tfrac{1}{2}) + A_1 f(+\tfrac{1}{2})$$

is exact for all degree at most one polynomials. (I.e. for all linear functions.)

(b) Show that the rule generated in part **(a)** is *not* exact for degree two polynomials.

6. Recall that if $\ell(x)$ is the piecewise-linear interpolant of $f \in C^2[a,b]$ at equally-spaced points $x_0 = a < x_1 < x_2 < \cdots < x_n = b$, with spacing $h = (b-a)/n$, then

$$|f(x) - \ell(x)| \le \frac{Mh^2}{8}$$

for all $x \in [a,b]$, where $M = \max_{x\in[a,b]} |f''(x)|$. Find n so that the error is at most 2×10^{-4} in using such equally-spaced linear interpolation for $f(x) = e^{-x}$ on $[a,b] = [0,2]$.

7. Do two steps of the Euler method, with step size $h = 1$, on the ODE IVP

$$y' = t - y, \qquad y(0) = 1.$$

8. (a) Sketch one step of the midpoint method for the general ODE IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

where $t_{k+1} - t_k = h$ is the step size. (Hints: *Your sketch will have t and y axes. Show the current iterate* (t_k, y_k) *and all the locations where a slope is computed. Show how to compute the new iterate* y_{k+1}*.*)

(b) Show that the midpoint method is exact when solving the ODE IVP

$$y' = 2t - 8, \qquad y(2) = 3.$$

9. Suppose the IEEE 754 standard for floating point representations had a 9 bit version:

s	e_1	e_2	e_3	b_1	b_2	b_3	b_4	b_5

representing the number

$$x = (-1)^s \, (1.b_1 b_2 b_3 b_4 b_5)_2 \; 2^{(e_1 e_2 e_3)_2 - 3_{10}}$$

Note the exception cases:

- exponent bits $(000)_2$ define the number zero or subnormal numbers
- exponent bits $(111)_2$ define the other exceptions: $\pm\infty$ and NaN (*…ignore the details*)

(a) What is the largest real number that this system can represent? (*State the number in decimal notation and show the bits.*)

(b) What is the value of "machine epsilon" in this system? (*State the number in decimal notation.*)

10. Suppose we want to use Taylor's theorem to compute values of $\sin x$ for $|x| < 0.5$ to an accuracy of 10^{-3}. Use the Taylor theorem with remainder to determine how many terms, i.e. what n, is needed to do this.

11. Solve the following system of linear equations by Gauss elimination with partial pivoting and back substitution. Show your steps.

$$2x_1 + 2x_2 = 6$$
$$4x_1 - 3x_2 = -2.$$

12. The high-level view of the Gauss elimination with partial pivoting algorithm is that, given a linear system

$$A\mathbf{x} = \mathbf{b},$$

it computes matrices P, L, U so that $PA = LU$. What properties do these matrices have? (*Write at least two complete sentences.*) Then explain how to solve the linear system, indicating how much work is required at each stage. (*Write at least two complete sentences.*)

13. (a) Write a MATLAB algorithm for multiplying a square $n \times n$ matrix A by an $n \times 1$ column vector $\mathbf{v}$. In particular, fill in the rest of the function below to compute

$$\mathbf{z} = A\mathbf{v}.$$

I have written the first line to get n. You may assume all sizes of the inputs are correct; there is no need to check these sizes. Do not use matrix-vector multiplication inside this routine; pretend that we are writing this for the first time and use `for` loops.

```
function z = mattimesvec(A,v)
% MATTIMESVEC multiplies A by v and gives z

n = length(v);
```

(b) Count the floating point operations in the above algorithm.

TABLE 10.3
Quadrature formulas and their errors.

Method	Approximation to $\int_a^b f(x)\,dx$	Error
Trapezoid rule	$\frac{b-a}{2}[f(a) + f(b)]$	$-\frac{1}{12}(b-a)^3 f''(\eta),\ \eta \in [a, b]$
Simpson's rule	$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$	$\frac{1}{2880}(b-a)^5 f^{(4)}(\xi),\ \xi \in [a, b]$
Composite trapezoid rule	$\frac{h}{2}[f_0 + 2f_1 + \ldots + 2f_{n-1} + f_n]$	$O(h^2)$
Composite Simpson's rule	$\frac{h}{6}[f_0 + 4f_{1/2} + 2f_1 + \ldots + 2f_{n-1} + 4f_{n-1/2} + f_n]$	$O(h^4)$

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