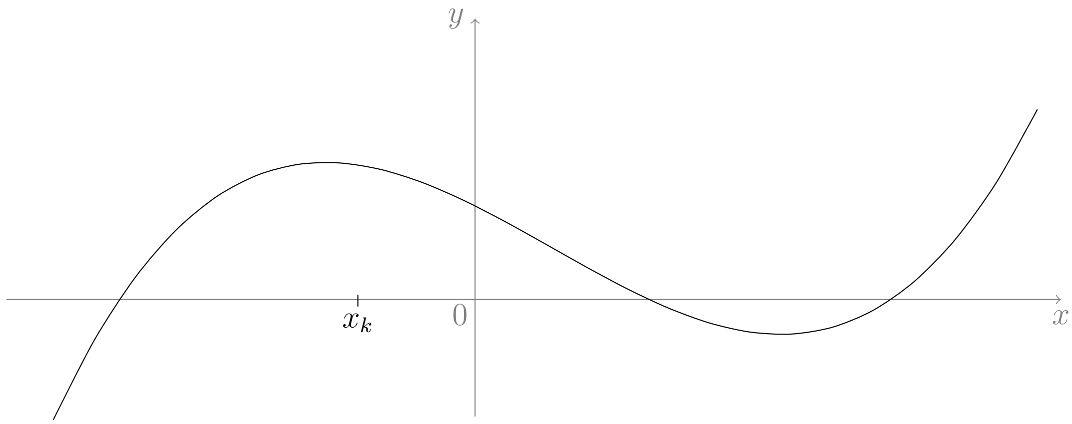
Name: ________________________________

Math 310 Numerical Analysis (Bueler) Thursday, 17 October 2019

Midterm Exam

In class. No book, electronics, or notes. 95 minutes maximum. 115 points possible.

1. (a) *(5 pts)* A differentiable function $f(x)$ and an iterate x_k are shown on the axes below. Sketch, with appropriate labeling, how Newton's method determines the next iterate x_{k+1}.



(b) *(10 pts)* Suppose $\ell(x)$ is the linearization of $f(x)$ at x_k. Give a formula for $\ell(x)$ and then a formula for where it crosses the x-axis. Write the result as Newton's method in the box below.

$x_{k+1} =$

2. **(a)** (*10 pts*) Consider the equation $x^3 - 3x + 1 = 0$ and suppose $a_0 = -1$ and $b_0 = 1$ is a bracket. Apply two steps of the bisection method, reporting the bracket at the end of each step.

(b) (*10 pts*) For the same equation as in **(a)**, with $x_0 = -1$ and $x_1 = 1$, apply one step of the secant method.

3. (*15 pts*) Solve the following system by Gauss elimination and back-substitution:

$$
\begin{aligned}
3x_1 + x_2 + 2x_3 &= 11 \\
3x_1 + 4x_2 + x_3 &= 14 \\
-3x_1 + 5x_2 - 2x_3 &= 1
\end{aligned}
$$

Show your steps in an organized way. It must be clear that you are following the algorithm we considered in class. (*Hints: Pivoting is* not *requested. The numbers are integers at every stage if you follow the algorithm.*)

4. (*5 pts*) Suppose we have used Gauss elimination with partial pivoting to factor some matrix A, so that $PA = LU$. Here P is a known permutation matrix, L is a known lower-triangular matrix, and U is a known upper-triangular matrix. Explain how to use this factorization to easily solve $A\mathbf{x} = \mathbf{b}$, assuming $\mathbf{b}$ is also given, and identify what algorithms are needed.

5. (*10 pts*) Consider lower-triangular matrices with unit diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{2,1} & 1 & 0 & & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ \ell_{n,1} & \ell_{n,2} & \dots & \ell_{n,n-1} & 1 \end{bmatrix}$$

Write a MATLAB/OCTAVE code, or a complete pseudocode, to solve systems $L\mathbf{y} = \mathbf{b}$, with L in the above form, by forward substitution.

```
function y = forwardsub(L,b)
```

6. Suppose that the IEEE standard for floating point representation discussed in class had an 8 bit version. It might look like this:

s	e_1	e_2	e_3	b_1	b_2	b_3	b_4

These 8 bits would represent the number

$$x = (-1)^s \, (1.b_1 b_2 b_3 b_4)_2 \, \times \, 2^{(e_1 e_2 e_3)_2 - 3}.$$

However, normal numbers would not use exponents $(000)_2$ nor $(111)_2$, which have special uses.

(a) (*5 pts*) What is the representation of 1 (one) in this system? (*Give all the bits.*)

(b) (*5 pts*) What is the largest number that this system can represent?

(c) (*5 pts*) What is the value of "machine epsilon" in this system?

(d) (*3 pts*) How would zero be represented? (*Give all the bits.*)

(e) (*2 pts*) Give the bits of a nonzero subnormal number. (*Just pick one.*)

7. (a) (*5 pts*) Find all the fixed points x_* of $\varphi(x) = \frac{1}{4}x^2 + \frac{3}{4}x - \frac{1}{2}$. (*Hint: There are two.*)

(b) (*10 pts*) Recall Theorem 4.5.1:

> *Theorem.* Assume that $\varphi \in C^1$ and $|\varphi'(x)| < 1$ in some interval $[x_* - \delta, x_* + \delta]$ around a fixed point x_* of φ. If x_0 is in this interval then the fixed point iteration converges to x_*.

For the same function $\varphi(x)$ as in part **(a)**, will the iteration

$$x_{k+1} = \varphi(x_k)$$

converge to each fixed point x_* for all x_0 near x_*? (*Hint: Consider each x_* in turn.*)

8. **(a)** *(5 pts)* State Taylor's theorem with remainder in the $n = 2$ case. Be sure to include the assumptions about the function $f(x)$.

Theorem.

(b) *(5 pts)* Using the Theorem above, find the quadratic (degree two) Taylor polynomial for $f(x) = \sqrt{x}$ using basepoint $a = 4$.

(c) *(5 pts)* The result of **(b)** is a polynomial $P_2(x)$ such that $f(x) \approx P_2(x)$. Use the Theorem in **(a)** to estimate the size of the error $|P_2(x) - f(x)|$ for all x in the interval $[3, 5]$.

Extra Credit. (*3 pts*) Do one step of Newton's method to find the first-quadrant intersection of the circle $x^2 + y^2 = 4$ and the graph $y = e^x$. Start from $(x_0, y_0) = (1, 2)$, which is not too far from the intersection.

BLANK SPACE BLANK SPACE