Matrix Norm Essentials

- All matrix norms have vector norm properties:
  - $\|A\| \geq 0$ and $\|A\| = 0 \implies A = 0$
  - $\|A + B\| \leq \|A\| + \|B\|$
  - $\|\alpha A\| = |\alpha| \, \|A\|$

- Induced matrix norms arise from vector norms. For $A \in \mathbb{C}^{m \times n}$,

$$\|A\| = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} = \max_{\|x\|_{(n)}=1} \|Ax\|_{(m)}$$

- This follows directly from the definition of an induced norm:
  - $\|Ax\|_{(m)} \leq \|A\| \|x\|_{(n)}$

- There are really only four matrix norms to know:

$$\|\cdot\|_1, \quad \|\cdot\|_2, \quad \|\cdot\|_\infty, \quad \|\cdot\|_{\text{Fro}}$$

  - 3 are induced from vector norms: $1, 2, \infty$
  - 3 have easy-to-compute formulas: $1, \infty, \text{Fro}$
  - `norm(A,1|Inf|"fro")` are fast in Matlab, `norm(A,2)` is slow

- Induced norms (*and* Frobenius) have a multiplicative property:
  - $\|AB\| \leq \|A\| \|B\|$

- Induced norms (*and* Frobenius) satisfy $\rho(A) \leq \|A\|$.
  - Recall $\rho(A) = \max |\lambda|$ where λ is an eigenvalue.
  - However, $\rho(A) \ll \|A\|$ is common. The norm can be a very conservative, that is, too large, estimate of $\rho(A)$.

- The $\|\cdot\|_2$ norm is best for Euclidean ideas and hermitian/normal matrices. Reasons:
  - $\|QA\|_2 = \|A\|_2$ if Q is unitary ($Q^*Q = I$).
  - Largest singular value: $\sigma_1(A) = \|A\|_2$.
  - If $A^* = A$ then $\rho(A) = \|A\|_2$.

- Iteration fact:

$$v, Av, A^2v, \ldots \text{ converges for all } v$$
$$\text{if and only if } \rho(A) < 1.$$

  - Thus if $\|A\| < 1$ then convergence.
  - But not conversely!