Matrix Norm Essentials

- Matrix norms have vector norm properties:
  - $\|A\| \geq 0$ and $\|A\| = 0 \implies A = 0$
  - $\|A + B\| \leq \|A\| + \|B\|$
  - $\|\alpha A\| = |\alpha| \, \|A\|$

- There are really only four norms to know:

  $$\| \cdot \|_1, \quad \| \cdot \|_2, \quad \| \cdot \|_\infty, \quad \| \cdot \|_{\text{Frob}}$$

  - 3 are induced from vector norms: $1, 2, \infty$
  - 3 have easy-to-compute formulas: $1, \infty, \text{Frob}$

- Induced norms (and Frobenius) have an additional multiplicative property:
  - $\|AB\| \leq \|A\| \|B\|$

- Induced norms (and Frobenius) satisfy $\rho(A) \leq \|A\|$.
  - Recall $\rho(A) = \max |\lambda|$ where λ is an eigenvalue.
  - However, $\rho(A) \ll \|A\|$ is common. The norm can be a very conservative estimate of $\rho(A)$.

- The $\| \cdot \|_2$ norm is best for Euclidean ideas and hermitian/normal matrices. Reasons:
  - $\|QA\|_2 = \|A\|_2$ if Q is unitary ($Q^*Q = I$).
  - Largest singular value: $\sigma_1(A) = \|A\|_2$.
  - If $A^* = A$ then $\rho(A) = \|A\|_2$.

- Iteration

  $$v, Av, A^2v, \ldots \text{ always converges}$$

  $$\text{if and only if } \rho(A) < 1.$$

  - Thus if $\|A\| < 1$ then convergence.
  - But not conversely!