 Not to be turned in!

Worksheet: Proving backward stability.

Recall axiom (13.5): for each $x \in \mathbb{R}$ there is a real value ϵ so that $\text{fl}(x) = x(1+\epsilon)$ and $|\epsilon| \leqslant \epsilon_{\text{machine}}$.
Recall axiom (13.7): for all $x, y \in \mathbb{R}$, and for operations $* = +, -, \times, \div$, there is a real value ϵ so that $x \circledast y = (x * y)(1+\epsilon)$ and $|\epsilon| \leqslant \epsilon_{\text{machine}}$.

A *problem* is a function $f : X \to Y$, where X and Y are normed vector spaces. An *algorithm* is another function $\tilde{f} : X \to Y$. On computers for which axioms (13.5) and (13.7) are true, we say the algorithm is *backward stable* for some input x if

$$\boxed{\tilde{f}(x) = f(\tilde{x})} \quad \text{for some } \tilde{x} \text{ satisfying} \quad \boxed{\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon_{\text{machine}})}.$$

In each of the following exercises, if I have not done it already, start by precisely identifying the problem and the algorithm. Then prove backward stability.

1. Prove that the obvious algorithm $\tilde{f}(x) = \text{fl}(x_1) \oplus\!\!\!\!\div\, \text{fl}(x_2)$ for dividing real numbers ($f(x) = x_1 \div x_2$) is backward stable. Assume $x_2 \neq 0$.

2. Prove that the obvious algorithm for raising a real number to the third power is backward stable.

3. Let $x \in \mathbb{R}^3$ be fixed. Prove that the obvious computer algorithm for computing the inner product a^*x, for $a \in \mathbb{R}^3$, is backward stable.