Assignment 8

Due Friday 10 November 2023, at the start of class

Please read Lectures 13,14,15,16,17 in the textbook *Numerical Linear Algebra* by Trefethen and Bau. This Assignment covers backward stability. Always assume axioms (13.5) and (13.7).

DO THE FOLLOWING EXERCISES from Lecture 14:

- **Exercise 14.1** *Do parts* (a), (b), (c), (f), *and* (g) *only.*
- **Exercise 14.2**

DO THE FOLLOWING EXERCISES from Lecture 15:

- **Exercise 15.1** *Do parts* (a), (b), (c), *and* (d) *only.*
- **Exercise 15.2**

DO THE FOLLOWING ADDITIONAL EXERCISES.

P17. *The goal of this exercise is to show that the usual matrix-vector multiplication algorithm is backward-stable when we regard the* matrix *as the input; see part* **(c)**. *For simplicity, please assume all entries are real numbers. Always assume axioms (13.5) and (13.7) hold.*

(a) Fix $x \in \mathbb{R}^1$. Show that if the problem is scalar multiplication, $f(a) = ax$ for $a \in \mathbb{R}^1$, then the obvious algorithm $\tilde{f}(a) = \text{fl}(a) \otimes \text{fl}(x)$ is backward-stable.

(b) Fix $x \in \mathbb{R}^2$, a column vector. Show that if $a \in \mathbb{R}^2$, a column vector, then the obvious algorithm $\tilde{f}(a) = \text{fl}(a_1) \otimes \text{fl}(x_1) \oplus \text{fl}(a_2) \otimes \text{fl}(x_2)$ for the inner product problem $f(a) = a^\top x$ is backward-stable. (*Hint. You must choose a vector norm to finish the proof.*)

A proof by induction extends part **(b)** *to show that the obvious inner product algorithm is backward-stable in any dimension; see Example 15.1. From now on you can assume it is true.*

(c) Fix $x \in \mathbb{R}^n$. Show that if $A \in \mathbb{R}^{m \times n}$ then the obvious algorithm $\tilde{f}(A)$ for the product problem $f(A) = Ax$ is backward-stable. (*Hints. Express Ax using inner products. Do not bother with scalar entries of A or x. You must pick a vector norm and an induced norm.*)

(d) Fix $A \in \mathbb{R}^{m \times n}$. Explain in at most 8 sentences why the obvious algorithm $\tilde{f}(x)$ for the problem $f(x) = Ax$ is generally ***not*** backward-stable. However, this result depends on dimension. In fact, for what m, n is this $\tilde{f}(x)$ backward-stable? (*Hints. The algorithm is the same as in part* **(c)**, *but the input is x. Use what we know for inner products.*)

Extra Credit. Is the standard algorithm for matrix-matrix multiplication backward-stable? Regard the problem with maximum flexibility, that is, as treating *both* factors as inputs: $F(A, B) = AB$ for $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$. Explain in at most 10 sentences. (*Hints. The answer depends on dimensions. Be precise and consider different cases in order to get full credit.*)