Assignment 7

Due Wednesday 1 November 2023 (*revised*), at the start of class

Please read Lectures 11,12,13,14,15 in the textbook *Numerical Linear Algebra* by Trefethen and Bau. This Assignment covers least squares, conditioning, and floating point.

DO THE FOLLOWING EXERCISES from Lecture 11:

- **Exercise 11.3**

DO THE FOLLOWING EXERCISES from Lecture 13:

- **Exercise 13.2** *Do parts* (a) *and* (b) *only.*

DO THE FOLLOWING ADDITIONAL EXERCISES.

P13. Suppose A is a 100×100 matrix with $\|A\|_2 = 20$ and $\|A\|_F = 21$. Give the sharpest possible lower bound on the 2-norm condition number of A. (*Hint. Write everything in terms of singular values, and then think about best cases for* $\kappa_2(A)$.)

P14. For each problem, compute the absolute condition number $\hat{\kappa}$ and the relative condition number κ; generally both formulas will involve x.[1] Choose the most convenient norm, but make your choice explicit.[2] Comment on when the problem is well-conditioned or ill-conditioned; generally this answer also depends on x.

a) Compute x^3 for $x > 0$.
b) Compute $\cos x$ for real x.
c) For $x \in \mathbb{C}^2$ compute $x_1 x_2$, the product of the entries.
d) Fix $a \in \mathbb{R}^m$, a column vector. Compute the inner product $a^* x$ for $x \in \mathbb{R}^m$.

P15. Consider the polynomial

$$\begin{aligned} p(x) &= (x-3)^{10} \\ &= x^{10} - 30x^9 + 405x^8 - 3240x^7 + 17010x^6 - 61236x^5 \\ &\quad + 153090x^4 - 262440x^3 + 295245x^2 - 196830x + 59049 \end{aligned}$$

(a) Por $x = \texttt{2.85:0.01:3.15}$, plot $p(x)$ by evaluating it via its coefficients $1, -30, 405, \ldots$

(b) Plot $p(x)$ again on the same interval and same graph, using expression $(x-3)^{10}$.

(c) In two or three sentences, compare and contrast the bad behavior here with the ill-conditioning phenomenon in Example 12.5 on page 92, i.e. Wilkinson's example.

[1] You can use formulas (12.3) and (12.6) without justification.

[2] For **a)** and **b)** just use absolute values for the norm.

P16. *This is a reading assignment. Actually read it! It's good.*

Please read the following 12 page encyclopedia entry:

> L. N. Trefethen, *Numerical Analysis*, in W. T. Gowers, editor, Princeton Companion to Mathematics, Princeton U. Press, 2008.
> people.maths.ox.ac.uk/trefethen/NAessay.pdf

Answer the following questions with a sentence or two at most:

(i) Give a one-sentence version of Trefethen's definition of "numerical analysis."
(ii) Is analysis of rounding errors the main business of numerical analysis? If not, what is?
(iii) Gaussian elimination with pivoting is a matrix factorization. State it.
(iv) Trefethen refers to Householder triangularization, Algorithm 10.1 in our textbook, as "QR factorization". But then what does the "QR algorithm" do?
(v) What is the "central dogma" of numerical linear algebra?
(vi) Fill the blank: "The discovery of _____ came quickly, but its theoretical analysis has proved astonishingly hard."
(vii) What is the "the biggest unsolved problem in numerical analysis"?