Review Topics
for in-class **Midterm Quiz 2** on
Wednesday 12 November 2025

The second Midterm Quiz will cover Lectures 10, 11, 12, 13, 14, 15, 17 in TREFETHEN & BAU. The problems will be of these types: **state definitions and axioms**, state theorems, state algorithms (as pseudocode or MATLAB), describe or illustrate geometrical ideas, basic applications of theorems, quick calculations, prove simple theorems/corollaries.

Definitions and Axioms. Know how to define, and how to use:

- residual $r = b - Ax$ for overdetermined "$Ax = b$"
- problem and problem instance (as defined in Lecture 12)
- Jacobian of a differentiable function (i.e. problem)
- absolute condition number $\hat{\kappa} = \hat{\kappa}_f(x)$ of a problem instance
- relative condition number $\kappa = \kappa_f(x)$ of a problem instance
- condition number of a square matrix: $\kappa(A) = \|A\|\,\|A^{-1}\|$
- idealized floating point system $\mathbb{F}$:
$$x = \pm \frac{m}{\beta^{t-1}} \beta^e, \quad \beta^{t-1} \le m \le \beta^t - 1$$
- function $f\ell : \mathbb{R} \to \mathbb{F}$
- axiom (13.5): $f\ell(x) = x(1 + \epsilon)$ where $|\epsilon| \le \epsilon_{\text{machine}}$
- axiom (13.7): $x \circledast y = (x * y)(1 + \epsilon)$ where $|\epsilon| \le \epsilon_{\text{machine}}$
- big O: $f(t) = O(g(t))$ as $t \to 0$ means there is $C \ge 0$ so that $|f(t)| \le C|g(t)|$
- big O: $\phi(n) = O(\psi(n))$ as $n \to \infty$ means there is $C \ge 0$ so that $|\phi(n)| \le C|\psi(n)|$
- backward stability of an algorithm
- stability of an algorithm

Background Definitions. The following may be needed somewhere, but I won't literally ask for the definition: matrix-vector product, matrix-matrix product, inner product, outer product, triangular matrix, unitary matrix, projector, orthogonal projector, $\|\cdot\|_p$ for vectors, induced matrix norm, $\|\cdot\|_F$, eigenvalue, eigenvector, singular value

Constructions. Know the properties of the objects in each construction below. Be able to use the construction in simple cases.

- reduced QR decomposition, wherein $\hat{Q}$ is basis for range(A): $A = \hat{Q}\hat{R}$
- orthogonal projector onto range(A): $P = \hat{Q}\hat{Q}^* = A\,(A^*A)^{-1}\,A^*$
- Householder reflector: $F = I - 2qq^* = I - 2vv^*/(v^*v)$
- linear system actually solved for overdetermined "$Ax = b$": $Ax = Pb$
- normal equations for overdetermined "$Ax = b$": $A^*Ax = A^*b$

Algorithms. Be able to state these algorithms, including the amount of work to leading order.

- Alg. 10.1: Householder triangularization for QR
- Alg. 10.2: compute Q^*b after Householder triangularization
- Alg. 11.1: solve normal equations for least squares on overdetermined "$Ax = b$"
  - just know that you use Cholesky … what that *is* will not be on Midterm
- Alg. 11.2: QR for least squares on overdetermined "$Ax = b$"
- Alg. 11.3: SVD for least squares on overdetermined "$Ax = b$"
  - *how* SVD is computed will not be on Midterm
- Alg. 16.1: solve $Ax = b$, A invertible, via QR factorization
- Alg. 17.1: back substitution for $Rx = b$ where R is upper-triangular and invertible

Facts/Formulas/Theorems. Know as facts. Be able to use in a context.

- for $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, A has full rank if and only if A^*A is nonsingular
- if P is an orthogonal projector then $I - 2P$ is unitary
- if A is square then $\kappa_2(A) = \sigma_1/\sigma_m$
- if $f(x)$ is a differentiable problem then $\hat{\kappa}(x) = \|J(x)\|$
- if $f(x)$ is a differentiable problem then $\kappa(x) = \dfrac{\|J(x)\|}{\|f(x)\|/\|x\|} = \dfrac{\|J(x)\|\|x\|}{\|f(x)\|}$
- Theorem 15.1: if $f(x)$ is a problem with condition number $\kappa(x)$, and if $\tilde{f}(x)$ is a backward stable algorithm for it, then $\dfrac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\epsilon_{\text{machine}})$
- Theorem 17.1: back substitution is backward stable

Ideas. Please be comfortable with the following ideas! Some ideas correspond to theorems, but others are just a perspective or paradigm.

- L10: Householder is "orthogonal triangularization". (Gram-Schmidt is "triangular orthogonalization" and Gauss elimination is "triangular triangularization".)
- L11: If A is a tall matrix, $\kappa(A^*A)$ is the square of $\kappa(A)$, which is why the normal equations might have an inaccurate solution in floating point.
- L12: Theorems 12.1, 12.2: All problems associated to linear system $Ax = b$ have $\kappa(A) = \|A\|\|A^{-1}\|$ as the relative condition number, or as a bound on it.
- L13: The most-useful definition of $\epsilon_{\text{machine}}$ is as the smallest number so that axioms (13.5) and (13.7) hold for your computer.
- L14: Inevitably the constant in your proof of backward stability or stability will depend on the vector/matrix dimension, i.e. m or n.
- L15: The usual inner product algorithm is backward stable. Not so for the outer product. These facts are instances of the (informal) idea that backward stability only occurs when the problem input has large enough dimension to assign blame to it.