Review Topics for in-class Midterm Quiz 1 on Wednesday 8 October 2025

The first Midterm Quiz will cover Lectures 1, 2, 3, 4, 5, 6, 7, and 8 in Trefethen & Bau. The problems will be of these types: state definitions, state theorems, state algorithms (as pseudocode or Matlab), describe or illustrate geometrical ideas, apply theorems in basic situations, do quick calculations, and prove simple theorems/corollaries.

Definitions. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose
- inner product; outer product
- unitary
- $\| \cdot \|_p$ of a vector in $\mathbb{C}^m$ for any $1 \leq p \leq \infty$
- induced matrix norm $\| \cdot \|$
- Frobenius matrix norm $\| \cdot \|_F$
- eigenvalue; eigenvector
- singular value
- projector; orthogonal projector

Matrix Factorizations and Constructions. Know the properties of the factors in each factorization below. (*For example, $\hat{U}$ has orthonormal columns and is of same size as A in the $m \geq n$ case of the reduced SVD factorization $A = \hat{U}\hat{\Sigma}V^*$.*) Assume $A \in \mathbb{C}^{m \times n}$ unless otherwise stated. Be able to use the factorization in simple calculations. Be able to compute the factorization by hand in small and simple cases.

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \geq n$: $A = \hat{U}\hat{\Sigma}V^*$
- full QR, $m \geq n$: $A = QR$
- reduced QR, $m \geq n$: $A = \hat{Q}\hat{R}$
- eigenvalue, $m = n$: $A = X\Lambda X^{-1}$ *← not always possible!*
- orthogonal projector onto range(A) if $m \geq n$ and A full rank:

$$P = \hat{U}\hat{U}^* = \hat{Q}\hat{Q}^* = A\,(A^*A)^{-1}\,A^*$$

Algorithms. Be able to state these algorithms, including the amount of work to leading order. For "how to solve $Ax = b$", there are high-level 4- and 3-step procedures, respectively.

- matrix-vector product
- matrix-matrix product
- Alg. 7.1: classical Gram-Schmidt for $\hat{Q}\hat{R}$
- Alg. 8.1: modified Gram-Schmidt for $\hat{Q}\hat{R}$
- how to solve $Ax = b$ when invertible matrix A has known SVD $A = U\Sigma V^*$
- how to solve $Ax = b$ when invertible matrix A has known QR decomposition $A = QR$

Facts and Formulas. Know as facts. Be able to prove unless otherwise stated.

- if A, B are invertible matrices then $(AB)^{-1} = B^{-1}A^{-1}$
- if A, B are matrices so that AB is defined then $(AB)^* = B^*A^*$
- Cauchy-Schwarz: $|x^*y| \leq \|x\| \, \|y\|$ [*proof not required*]
- invariance of $\| \cdot \|_2$ and $\| \cdot \|_F$ matrix norms under unitaries
- $\|A\|_2 = \sigma_1$ and $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$
- for $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, A has full rank if and only if A^*A is nonsingular
- rank(A) is number of nonzero singular values (in exact arithmetic)
- for square matrices, $|\det(A)| = \prod_{i=1}^{m} \sigma_i$
- the singular values of A are the square roots of the eigenvalues of A^*A
- if P is an (orthogonal) projector then $I - P$ is an (orthogonal) projector
- if P is an orthogonal projector then $I - 2P$ is unitary

Ideas. Please be comfortable with the following ideas! Some ideas correspond to theorems, but otherwise it is just a perspective or paradigm.

- L1 and L2: how to think about Ax, $A^{-1}b$, Qx, Q^*b
- L4: the image of the unit sphere under any $m \times n$ matrix is a hyperellipsoid
- L5: sums like this are optimal approximations of A (in what sense?):

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

- L6: given A, the orthogonal projector onto range(A) is constructable; see formulas above
- L8: the leading-order number of operations in some algorithm can be counted by geometric arguments