Name: SOLUTIONS

Midterm Quiz 1

In-class or proctored. No book, notes, electronics, calculator, internet access, or communication with other people. 100 points possible.
<u>65 minutes</u> maximum!

1. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(a) (*5 pts*) What are the eigenvalues of A? (Give some brief explanation, or show work.)

$\lambda_1 = \lambda_2 = 1$ because the eigenvalues of triangular matrices are on the diagonal

[also: $p(\lambda) = \det(\lambda I - A) = (\lambda - 1)^2$ has root $\lambda_1 = \lambda_2 = 1$]

(b) (*10 pts*) What is $\|A\|_2$? (*Hint.* Start as though you are finding singular values?)

$$B = A^* A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(\lambda I - B) = (\lambda - 1)(\lambda - 2) - 1 = \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \sigma_i^2$$

key idea

thus

$$\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$$

so

$$\|A\|_2 = \sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$$

2. Suppose A is an invertible $m \times m$ matrix which has SVD $A = U\Sigma V^*$. Let σ_j denote the ordered singular values of A.

(a) (*7 pts*) Write A^{-1} in terms of U, Σ, and V, and simplify.

$$A^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^*$$

(b) (*8 pts*) From **(a)**, explain why $\|A^{-1}\|_2 = 1/\sigma_m$.

$$\|A^{-1}\|_2 = \|V\Sigma^{-1}U^*\|_2 = \|\Sigma^{-1}\|_2$$

$\|\cdot\|_2$ is unitarily invariant

$$= \left\|\begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_m \end{bmatrix}\right\|_2$$

$= 1/\sigma_m$ because $\|\cdot\|_2$ of diagonal matrices is largest entry

Extra Credit A. (*3 pts*) Continuing problem **2** above, construct an SVD of A^{-1} from the SVD of A. (*Hint.* More subtle. Be careful with the ordering required for SVD.)

$$A^{-1} \overset{(a)}{=} V\Sigma^{-1}U^* = (VP)(P\Sigma^{-1}P)(PU^*)$$

reverses order of rows (on left) and columns (on right)

where $P = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & ⋰ & & \\ 1 & & & \end{bmatrix}$ satisfies $P^2 = I$, $P^* = P$

$\tilde{U} = VP$, $\tilde{\Sigma} = P\Sigma^{-1}P = \begin{bmatrix} 1/\sigma_m & & \\ & \ddots & \\ & & 1/\sigma_1 \end{bmatrix}$, $\tilde{V} = UP$

$$\Rightarrow A^{-1} = \tilde{U}\tilde{\Sigma}\tilde{V}^*$$

3. (*10 pts*) Show that if Q is unitary and λ is an eigenvalue of Q then $|\lambda| = 1$. (*Hint.* If z is a complex number then $|z|^2 = \bar{z}z$ where $\bar{z}$ is the conjugate.)

pf: Suppose $Qx = \lambda x$ for $x \neq 0$. Then

$$(Qx)^*(Qx) = (\lambda x)^* \lambda x = x^* \bar{\lambda} \lambda x = |\lambda|^2 \|x\|_2^2$$

but also

$$(Qx)^*(Qx) = x^* \underbrace{Q^* Q}_{= I \text{ since unitary}} x = x^* I x = \|x\|_2^2.$$

Thus $|\lambda|^2 \|x\|_2^2 = \|x\|_2^2$, so $|\lambda|^2 = 1$ since $x \neq 0$.

Then $|\lambda| = 1$. $\blacksquare$

4. (*10 pts*) Suppose $q \in \mathbb{C}^m$ is a unit vector. Show $F = I - 2qq^*$ is unitary.

pf:

$$\begin{aligned} F^*F &= (I - 2qq^*)^*(I - 2qq^*) \\ &= (I - 2(qq^*)^*)(I - 2qq^*) \\ &= (I - 2qq^*)(I - 2qq^*) \\ &= I - 2qq^* - 2qq^* + 4q\underbrace{q^*q}_{=\|q\|_2^2 = 1}q^* \\ &= I - 4qq^* + 4qq^* \\ &= I, \end{aligned}$$

so F is unitary. $\blacksquare$

5. (a) (*10 pts*) Let $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. Write a pseudocode or MATLAB code for the standard algorithm which computes the matrix-vector product Ax.

```
function b = matvec(A, x)
    b = 0
    for i = 1:m
        for j = 1:n
            b_i = b_i + a_ij * x_j
```

(or you can save one addition per b_i)

(b) (*5 pts*) Exactly how many floating point operations occur in the above algorithm?

$$m \cdot n \cdot 2 = \boxed{2mn}$$

outer loop

inner loop

op

$(= m \cdot (n \cdot 2 - 1) = 2mn - m$ if you saved above)

6. **(a)** (*7 pts*) Suppose $\hat{Q} \in \mathbb{C}^{m\times n}$ has orthonormal columns. The product $\hat{Q}^*\hat{Q}$ has a particularly simple form. State what the form is and prove it.

$$\hat{Q}^*\hat{Q} = I_{n\times n}$$

Pf: In the matrix product AB, the entry $(AB)_{ij}$ is (essentially) an inner product $(AB)_{ij} = a_i^* b_j$. In our case,

$$(\hat{Q}^*\hat{Q})_{ij} = q_i^* q_j = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

because the columns $\{q_i\}$ are ON. And those entries are the entries of I. $\boxtimes$

(b) (*8 pts*) Continuing with the same matrix $\hat{Q}$, show that $P = \hat{Q}\hat{Q}^*$ is an orthogonal projector.

$P^2 = P$ and $P^* = P$

Pf: $$P^2 = \hat{Q}\hat{Q}^*\hat{Q}\hat{Q}^* = \hat{Q}\underbrace{(\hat{Q}^*\hat{Q})}_{=I}\hat{Q}^*$$
$$= \hat{Q}\hat{Q}^* = P$$

$$P^* = (\hat{Q}\hat{Q}^*)^* = \hat{Q}^{**}\hat{Q}^*$$
$$= \hat{Q}\hat{Q}^* = P. \quad \boxtimes$$

7. (*10 pts*) By any by-hand method, compute a reduced QR decomposition of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad \text{since} \quad r_{11} = \|a_1\|_2 = \sqrt{2}$$

$$a_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow v_2 = a_2 - \underbrace{(q_1^* a_2)}_{(=r_{12} = 1/\sqrt{2})} q_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \end{bmatrix} \Rightarrow r_{22} = \|v_2\|_2 = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$q_2 = \frac{v_2}{r_{22}} = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \end{bmatrix}$$

$$\therefore \quad A = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix}$$

8. (*5 pts*) Suppose $A \in \mathbb{C}^{m\times n}$ and that $\|\cdot\|$ is a norm on $\mathbb{C}^n$ and that $\|\cdot\|'$ is a norm on $\mathbb{C}^m$. Define the *induced matrix norm* $\|A\|$. (*Full credit requires using the correct norms in the correct places.*)

or "max"

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|'}{\|x\|} = \sup_{\|x\|=1} \|Ax\|'$$

either form o.k.

you can say "$x \in \mathbb{C}^n$" for clarity

9. (*5 pts*) Suppose P is an orthogonal projector. Show $I - P$ is an orthogonal projector.

$$(I-P)^2 = I-P, \quad (I-P)^* = I-P$$

pf: $$(I-P)^2 = I - P - P + P^2 \underset{P^2=P}{=} I - 2P + P = I - P$$

$$(I-P)^* = I^* - P^* \underset{P^*=P}{=} I - P. \quad \blacksquare$$

Extra Creditβ. (*3 pts*) Suppose $A \in \mathbb{R}^{3\times 3}$, and suppose we know the value of the following norms: $\|A\|_2 = 10$, $\|A^{-1}\|_2 = 1$, $\|A\|_F = 11$. Find all singular values of A.

$$\|A\|_2 = 10 = \sigma_1$$

$$\|A^{-1}\|_2 \overset{2(b)}{=} 1/\sigma_3 = 1 \quad \therefore \sigma_3 = 1$$

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \sqrt{10^2 + \sigma_2^2 + 1} = 11$$

$$\therefore \quad \sigma_2^2 = 121 - 101 = 20$$

$$\therefore \quad \sigma_2 = \sqrt{20}$$