

## Review Topics

### for in-class Midterm Quiz 2 on Wednesday 15 November 2023

The second Midterm Quiz will cover Lectures 10, 11, 12, 13, 14, 15, 16, 17 in TREFETHEN & BAU. The problems will be of these types: **state definitions and axioms**, state theorems, state algorithms (as pseudocode or MATLAB), describe or illustrate geometrical ideas, basic applications of theorems, quick calculations, prove simple theorems/corollaries.

**Definitions and Axioms.** Know how to define, and how to use:

- residual (for overdetermined “ $Ax = b$ ”)
- problem (as defined in Lecture 12)
- absolute condition number  $\hat{\kappa}(x)$  and relative condition number  $\kappa(x)$  of a problem instance
- condition number of a square matrix:  $\kappa(A) = \|A\| \|A^{-1}\|$
- idealized floating point system  $\mathbb{F}$  and function  $f\ell(x)$
- axiom (13.5):  $f\ell(x) = x(1 + \epsilon)$  where  $|\epsilon| \leq \epsilon_{\text{machine}}$
- axiom (13.7):  $x \otimes y = (x * y)(1 + \epsilon)$  where  $|\epsilon| \leq \epsilon_{\text{machine}}$
- big O:  $f(t) = O(g(t))$  as  $t \rightarrow 0$  means there is  $C \geq 0$  so that  $|f(t)| \leq C|g(t)|$  for all  $t \dots$
- big O:  $\phi(n) = O(\psi(n))$  as  $n \rightarrow \infty$  means there is  $C \geq 0$  so that  $|\phi(n)| \leq C|\psi(n)| \dots$
- backward stability and stability of an algorithm

**Background Definitions.** The following may be needed somewhere, but I won’t literally ask for the definition: matrix-vector product, matrix-matrix product, inner product, outer product, triangular matrix, unitary matrix, projector, orthogonal projector,  $\|\cdot\|_p$  for vectors, induced matrix norm,  $\|\cdot\|_F$ , eigenvalue, eigenvector, singular value

**Constructions.** Know the properties of the objects in each construction below. Be able to use the construction in simple cases.

- orthogonal projector onto  $\text{range}(A)$ :  $P = \hat{Q}\hat{Q}^* = A(A^*A)^{-1}A^*$
- Householder reflector:  $F = I - 2qq^* = I - 2vv^*/(v^*v)$
- normal equations for overdetermined “ $Ax = b$ ”:  $A^*Ax = A^*b$

**Algorithms.** Be able to state these algorithms, including the amount of work to leading order.

- Alg. 10.1: Householder triangularization for  $QR$
- Alg. 11.1: solve normal equations for least squares on overdetermined “ $Ax = b$ ”
  - just know that you use Cholesky ... what that *is* will not be on Midterm
- Alg. 11.2: QR for least squares on overdetermined “ $Ax = b$ ”
- Alg. 11.3: SVD for least squares on overdetermined “ $Ax = b$ ”
  - *how* SVD is computed will not be on Midterm
- Alg. 16.1: solve  $Ax = b$ ,  $A$  invertible, via QR factorization
- Alg. 17.1: back substitution for  $Rx = b$  where  $R$  is upper-triangular and invertible

**Facts/Formulas/Theorems.** Know as facts. Be able to use in a context.

- for  $A \in \mathbb{C}^{m \times n}$  with  $m \geq n$ ,  $A$  has full rank if and only if  $A^*A$  is nonsingular
- if  $P$  is an orthogonal projector then  $I - 2P$  is unitary
- if  $A$  is square then  $\kappa_2(A) = \sigma_1/\sigma_m$
- if  $f(x)$  is a differentiable problem then  $\hat{\kappa}(x) = \|J(x)\|$
- if  $f(x)$  is a differentiable problem then  $\kappa(x) = \frac{\|J(x)\|}{\|f(x)\|/\|x\|} = \frac{\|J(x)\|\|x\|}{\|f(x)\|}$
- Theorem 15.1: if  $f(x)$  is a problem with condition number  $\kappa(x)$ , and if  $\tilde{f}(x)$  is a backward stable algorithm for it, then  $\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\epsilon_{\text{machine}})$
- Theorem 16.1: QR factorization is backward stable
- Theorem 16.2: Alg. 16.1 is backward stable
- Theorem 16.3: the solution  $\tilde{x}$  computed by Alg. 16.1 satisfies  $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\kappa(A)\epsilon_{\text{machine}})$
- Theorem 17.1: back substitution is backward stable

**Ideas.** Please be comfortable with the following ideas! Some ideas correspond to theorems, but others are just a perspective or paradigm.

- L10: Householder is “orthogonal triangularization”. (Gram-Schmidt is “triangular orthogonalization” and Gauss elimination is “triangular triangularization”.)
- L12: Theorems 12.1, 12.2: Problems associated to linear system  $Ax = b$  have  $\kappa(A) = \|A\|\|A^{-1}\|$  as the relative condition number, or as a bound on it.
- L13: The most useful definition of  $\epsilon_{\text{machine}}$  is as the smallest number so that axioms (13.5) and (13.7) hold for your computer.
- L14: It is o.k. for the constant in your proof of backward stability or stability to depend on the vector/matrix dimension, i.e.  $m$  or  $n$ .
- L15: The usual inner product algorithm is backward stability. Not so for the outer product. These facts are instances of the (informal) idea that backward stability only occurs when the problem input has large enough dimension to assign blame on it.