Name: ____________________________________

Math 614 Numerical Linear Algebra (Bueler) Wednesday 15 November 2023

Midterm Quiz 2

In-class or proctored. No book, notes, electronics, calculator, internet access, or communication with other people. 100 points possible.
<u>65 minutes</u> maximum!

1. (*10 pts*) Suppose $f : X \to Y$ is a problem, where X and Y are normed vector spaces. For $x \in X$, define the *relative condition number* of the problem:

$\kappa(x) =$

2. (*15 pts*) Our textbook TREFETHEN & BAU defines an idealized floating point system $\boldsymbol{F}$, also written $\mathbb{F}$. Define/describe it. (*Hints.* A floating point system is scientific notation based on a base β and a precision t. Both β and t are integers; what are their ranges? Now precisely describe the allowed fractions and exponents.)

3. **(a)** (*5 pts*) State axiom (13.5).

(b) (*5 pts*) State axiom (13.7).

4. (*15 pts*) Suppose $f : X \to Y$ is a problem and $\tilde{f} : X \to Y$ is an algorithm to compute (approximate) that problem on a computer satisfying axioms (13.5) and (13.7). Define what it means for the algorithm $\tilde{f}$ to be *backward stable* for the input $x \in X$.

5. (*7 pts*) Show that $(1 + O(t))(1 + O(t)) = 1 + O(t)$ as $t \to 0$.

6. (*7 pts*) Consider the problem (function) $f(x) = x^4$ on real numbers. Compute the absolute condition number $\hat{\kappa}(x)$ and the relative condition number $\kappa(x)$.

7. (*8 pts*) Suppose $A \in \mathbb{C}^{m\times m}$ is invertible, and that $b \in \mathbb{C}^m$. Explain, via major steps, how to use the QR factorization to solve the linear system $Ax = b$. How much work,[1] i.e. how many floating point operations, is required for each step?

8. (*8 pts*) Suppose $A \in \mathbb{C}^{m\times n}$ is full rank, and that $m \geq n$. Suppose $b \in \mathbb{C}^m$. Explain, via major steps, how to use the reduced SVD factorization to solve the overdetermined system "$Ax = b$" by least squares. How much work is required for each step?

[1] For problems **7** and **8**, use big-O notation to communicate the amount of work at leading order in m and/or n, as they go to infinity. You do not need to prove your big-O usage.

9. Suppose $x \in \mathbb{R}^2$ and that $f(x) = x_1^2 + x_2^2$.

(a) (*4 pts*) Write the obvious floating point algorithm for computing f, using notation $\mathrm{f}\ell(\cdot)$, $\oplus$, $\otimes$:

$\tilde{f}(x) =$

(b) (*8 pts*) Assuming a computer satisfying axioms (13.5) and (13.7), show that the above algorithm is backward stable. You may assume here, without proof, that $(1 + O(t))(1 + O(t)) = 1 + O(t)$ and $\sqrt{1 + O(t)} = 1 + O(t)$ as $t \to 0$.

Extra Credit (use space on blank pages). (*2 pts*) Assuming that the result in **9 (b)** above can be extended to $x \in \mathbb{R}^m$ for any m, argue that the obvious algorithm for computing the 2-norm of a vector is backward stable. Along the way you will need to describe, and briefly justify, the expected stability properties of a fifth arithmetic operation.

10. (*8 pts*) Suppose I invent a new way of solving linear systems which is even more stable than the Householder reflection QR method. The Bueler algorithm solves $Ax = b$, for $A \in \mathbb{C}^{m\times m}$ invertible and $b \in \mathbb{C}^m$, in a backward stablle manner, with numerical result $\tilde{x} \in \mathbb{C}^m$ satisfying $(A + \delta A)\tilde{x} = b$ where $\|\delta A\|_2/\|A\|_2 \le 30\log_{10}(m)\epsilon_{\text{machine}}$.[2] On a computer with $\epsilon_{\text{machine}} = 10^{-16}$, I apply the Bueler algorithm to solve a linear system for a certain matrix $A \in \mathbb{C}^{1000\times 1000}$ for which I know that the 2-norm condition number is $\kappa_2(A) = 10^9$. How many digits of accuracy will I have in the answer $\tilde{x}$? (*Hints.* Start by being clear on what is the problem. Apply big ideas precisely, but avoid little algebra.)

[2] That is, $\frac{\|\delta A\|_2}{\|A\|_2} = O(\epsilon_{\text{machine}})$ with constant $C = 30\log_{10}(m)$. This C is much smaller than for Householder QR.

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