Summary: Why Finite Difference Methods Work

Chapter 2 of the textbook (R. J. LeVeque, 2007) starts with the ODE BVP example

$$u''(x) = f(x), \qquad u(0) = \alpha, \quad u(1) = \beta.$$

The book constructs a practical finite difference (FD) numerical method for this example. Then the book explains why the numerical solution will converge to the exact solution as we refine the grid ($m \to \infty$ and $h \to 0$). The argument is "consistency + stability $\implies$ convergence," the useful direction of the Lax equivalence theorem*. This summary puts the argument on one page, with details suppressed. Note that stages 1 and 2 compute a numerical solution. Stages 3 and 4 are about why you get convergence.*

Stage 1. Apply scheme to linear DE. Choose the grid/mesh, including number of unknowns m and spacing $h > 0$. Apply your FD *discretization* or *scheme*; for each h you seek the values U_j^h in a vector $U^h \in \mathbb{R}^m$. Your scheme creates a *family* of matrices A^h and right-hand sides F^h, thus linear systems:

$$\begin{pmatrix} \text{differential equation (DE)} \\ \text{and boundary/initial conditions} \end{pmatrix} \qquad \to \qquad A^h U^h = F^h$$

Stage 2. Solve the scheme. Numerically solve the system of (linear) algebraic equations for a given h. Abstractly this is:

$$A^h U^h = F^h \qquad \to \qquad U^h = (A^h)^{-1} F^h$$

Stage 3. LTE and error equation. Let $\hat{U}_j^h = u(x_j)$ be the grid values of the exact solution $u(x)$ of your DE. (Note: $u(x)$ is generally unknown!) Define the *local truncation error* (LTE) as the residual from the scheme, when it is applied to the exact solution,

$$\tau^h = A^h \hat{U}^h - F^h.$$

A Taylor's theorem computation gives the *order of accuracy* p: $\|\tau^h\| = O(h^p)$. If $p > 0$ then the scheme is *consistent*. Define the *numerical error* $E^h = U^h - \hat{U}^h$. Subtract for the *error equation*: $A^h E^h = -\tau^h$.

Stage 4. Show stability to show convergence. Show *stability*: there is $C > 0$ so that $\|(A^h)^{-1}\| \le C$ for all $h > 0$. (Stability may be difficult to show!) Since A^h is invertible, the error equation has a solution: $E^h = -(A^h)^{-1}\tau^h$. Because $\|\tau^h\| = O(h^p)$, get *convergence* at rate p:

$$\|E^h\| = \| - (A^h)^{-1}\tau^h\| \le \|(A^h)^{-1}\| \|\tau^h\| \le C O(h^p) = O(h^p)$$