Von Neumann analysis: plug and chug

Section 9.6 of the textbook[1] *introduces von Neumann analysis, but without showing how people actually* do *it. This worksheet reveals the standard style. The textbook eventually says it clearly int the first sentences of section 10.5.*

Example. FTCS on heat equation. It is easiest to explain the idea relative to an example. Suppose we apply the FTCS scheme to the heat equation $u_t = Du_{xx}$ with constant diffusivity $D > 0$:

$$\frac{U_j^{n+1} - U_j^n}{k} = D\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} \tag{1}$$

To find what time steps $k > 0$ would be stable for a given spacing $h > 0$, von Neumann substituted

$$U_j^n = g(\xi)^n e^{ijh\xi} \tag{2}$$

into scheme (1). Here $\xi \in \mathbb{R}$ is the *wave number* for the spatial wave $e^{ijh\xi}$, in which $i = \sqrt{-1} \in \mathbb{C}$ (as usual). The scalar function $g(\xi)$ is called the *amplification factor* of the scheme. The spatial wave is complex, but it really is a wave, and for the interval $0 \le x \le 1$ the grid is $x_j = jh$ and thus

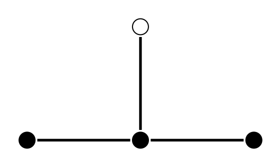
$$e^{ijh\xi} = e^{i\xi x_j} = \cos(\xi x_j) + i\sin(\xi x_j).$$

To understand stability we want to find $g(\xi)$. To compute it, substitute form (2) into scheme (1). Indices "$n+1$," "$j-1$," and "$j+1$" will turn into powers. Then use the properties of the exponential. After simplification and trigonometric identities—please do the details in Exercise 1 below—we get

$$g(\xi) = 1 - \frac{4Dk}{h^2}\sin^2\left(\frac{\xi h}{2}\right).$$

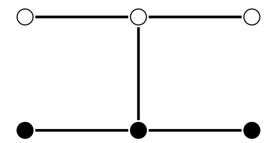
Absolute stability $|U_j^{n+1}| \le |U_j^n|$ corresponds to $|g(\xi)| \le 1$ for all $\xi \in \mathbb{R}$. For this scheme we get the condition $k \le \dfrac{h^2}{2D}$. The same condition is derived via MOL in section 9.3. In conclusion, the time step k must be very small when the spacing h is small.

Exercise 1. FTCS on heat equation. Label the stencil, and then fill in the above details.



[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

Exercise 2. Crank-Nicolson on heat equation. State the scheme. von Neumann it.



Exercise 3. Forward time and upwinding on the advection equation $u_t + au_x = 0$, $a > 0$ constant. State the scheme, draw and label a stencil, and do the analysis.

Exercise 4. CTCS (leapfrog) on the same advection equation. Same instructions.