Worksheet: Stable time-steps for 2nd-order ODE schemes

METHODS.[1] Consider these 3 ODE IVP methods for an autonomous ODE system $u' = f(u)$:

method	*book reference*	*formula*
TR: trapezoidal	(5.22)	$U^{n+1} = U^n + \frac{k}{2}\left(f(U^n) + f(U^{n+1})\right)$
EM: explicit midpoint	(5.30)	$U^{n+1} = U^n + kf\left(U^n + \frac{k}{2}f(U^n)\right)$
AB2: Adams-Bashforth	p. 132	$U^{n+2} = U^{n+1} + \frac{k}{2}\left(-f\left(U^n\right) + 3f\left(U^{n+1}\right)\right)$

All 3 methods have $O(k^2)$ local truncation error. Note EM is an explicit RK2 method.

PROBLEMS. Consider the following three linear ODE systems:

S1. $u' = Au$ where $u(t) \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & -5 \end{bmatrix}$$

S2. $u' = Au$ where $u(t) \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

S3. $u' = Au$ where $u(t) \in \mathbb{R}^{25}$ and A is the tridiagonal matrix shown in equation (2.10) for the $m = 25$ case (i.e. $h = 1/26$ case); this A approximates a second derivative in space

TASKS.

(a) For each of the one-step METHODS, find the stability function $R(z)$, for $z = k\lambda$, and write a MATLAB etc. code, using `contourf` or `imagesc` or similar as needed, to plot the region of absolute stability.

(b) For the AB2 METHOD, which is a multistep scheme, a slightly different approach is needed to plot the stability region. Following the approach of section 7.3, use the facts that $\rho(\zeta) = \zeta^2 - \zeta$, $\sigma(\zeta) = (3/2)\zeta - (1/2)$ to compute a quadratic polynomial $\pi(\zeta; z)$. Then you can either follow the approach of 7.6.1, or check the root condition at every point in a mesh and apply `contourf` etc. to plot the stability region.[2]

(c) For each PROBLEM, compute/approximate all of the eigenvalues λ_p of A.

(d) For each pair (METHOD, PROBLEM), determine the maximum absolutely-stable time step k_{stab}, or explain why there is no stability restriction on the time step ($k_{\text{stab}} = +\infty$). For $k_{\text{stab}} < \infty$ cases show a figure which has the relevant z-values ($z = k_{\text{stab}}\lambda_p$) plotted on top of the stability region.

(e) For each of the PROBLEMS, give expert advice: which METHOD is best and why? For this question think of various ways in which a given method does or does not preserve "qualities" relevant to the particular problem. Computational cost is a consideration too; assume your computer is slow. (*Hints. See section 7.5 and think about k_{stab} and k_{acc}. See also sections 8.3 and 8.4; A-stable and L-stable properties are relevant.*)

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[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

[2] Give this task a shot here on the worksheet. But it will not be on homework, and don't sweat it.