Summary: Why Finite Difference Methods Work

Chapter 2 of the textbook (R. J. LeVeque, 2007) starts with the ODE BVP example

$$u''(x) = f(x), \qquad u(0) = \alpha, \quad u(1) = \beta.$$

It constructs a practical finite difference (FD) numerical method on this example. Then it explains why the numerical solution will converge to the exact solution as we refine the grid ($m \to \infty$ and $h \to 0$). We will use the same basic "consistency + stability $\implies$ convergence" strategy on all problems, the Lax equivalence theorem. *This summary puts the whole strategy on one page, for linear DEs, with details suppressed.*

To use this as a worksheet: *Fill in the extra space, or the reverse side, with your details!*

Stage 1. Apply scheme to DE. Choose the grid/mesh, including number of unknowns m and spacing h. Apply your FD *discretization* or *scheme*; it creates a *family* of matrices $\{A^h\}$:

$$\begin{pmatrix}\text{differential equation (DE)}\\ \text{and boundary/initial conditions}\end{pmatrix} \qquad \implies \qquad A^h U^h = F^h$$

Stage 2. Solve the scheme. Numerical solution of the system of (linear) algebraic equations:

$$A^h U^h = F^h \qquad \implies \qquad U^h = (A^h)^{-1} F^h$$

Stage 3. LTE and error equation. Let $\hat{U}^h_j = u(x_j)$ be the grid values of the exact solution $u(x)$ of your DE. (You may not know $u(x)$!) Define the *local truncation error* (LTE) as the residual from the scheme, when it is applied to the exact solution:

$$\tau^h = A^h \hat{U}^h - F^h = O(h^p)$$

Taylor's theorem generates the *order of accuracy* p, and if $p > 0$ then the scheme is *consistent*. Defining the *numerical error* $E^h = U^h - \hat{U}^h$, get:

$$A^h E^h = -\tau^h \qquad \implies \qquad E^h = -(A^h)^{-1} \tau^h$$

Stage 4. Apply stability to show convergence. Show *stability*: there is $C > 0$ so that $\|(A^h)^{-1}\| \le C$ for all $h > 0$. (Stability may be difficult to show!) Because $\|\tau^h\| = O(h^p)$, get *convergence* at rate p:

$$\|E^h\| = \|-(A^h)^{-1}\tau^h\| \le \|(A^h)^{-1}\| \|\tau^h\| \le C O(h^p) = O(h^p)$$