Von Neumann analysis: plug and chug

Section 9.6 of the textbook (R. J. LeVeque, 2007) presents von Neumann analysis without emphasizing how people actually do *the analysis. This worksheet exercises the standard style.*[1]

Example. FTCS on heat equation. It is easiest to explain the idea relative to an example. Suppose we apply the FTCS scheme to the heat equation $u_t = Du_{xx}$ with constant diffusivity $D > 0$:

$$\frac{U_j^{n+1} - U_j^n}{k} = D\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} \tag{1}$$

To find what time steps $k > 0$ would be stable for a given spacing $h > 0$, von Neumann substituted

$$U_j^n = g(\xi)^n e^{ijh\xi} \tag{2}$$

into scheme (1), where $\xi \in \mathbb{R}$ is the *wave number* for the spatial wave $e^{ijh\xi}$, in which $i = \sqrt{-1} \in \mathbb{C}$ (as usual). The scalar function $g(\xi)$ is called the *amplification factor* of the scheme.

The spatial wave is complex, but it really is a wave. In fact, for the interval $0 \le x \le 1$ the grid is $x_j = jh$ and thus

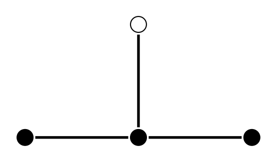
$$e^{ijh\xi} = e^{i\xi x_j} = \cos(\xi x_j) + i\sin(\xi x_j).$$

Now we want to find $g(\xi)$. To compute it, substitute form (2) into scheme (1). Indices "$n+1$," "$j-1$," and "$j+1$" will turn into powers. Then use the properties of the exponential. After simplification and trigonometric identities—do the details in Exercise 1—we get

$$g(\xi) = 1 - \frac{4Dk}{h^2}\sin^2\left(\frac{\xi h}{2}\right).$$

Absolute stability $|U_j^{n+1}| \le |U_j^n|$ corresponds to $|g(\xi)| \le 1$ for all $\xi \in \mathbb{R}$. For this scheme[2] we get the condition $k \le \frac{h^2}{2D}$, which tells us that the time step k must be very small if the spacing h is small.

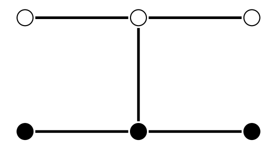
Exercise 1. FTCS on heat equation. Label the stencil. Then fill in the above details.



[1] The textbook eventually says it clearly in section 10.5.

[2] The same condition is derived via MOL in section 9.3, and then by von Neumann analysis in section 9.6.

Exercise 2. Crank-Nicolson on heat equation. Label the stencil. State the scheme. Do the analysis.



Exercise 3. FTCS on the advection equation $u_t + au_x = 0$ with a constant. Write down the scheme, draw and label a stencil, and do the analysis.

Exercise 4. CTCS (leapfrog) on advection equation. Again.