Assignment #7

Due Friday, 11 April 2025, at the start of class

Please read Chapters 6, 7, and 8 of the textbook (R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007). Within this material we are de-emphasizing multistep methods, so sections 5.9, 6.4, 7.3, 7.6.1, and 7.7 are all optional. However, full understanding of the other sections is expected; please actually set aside a bit of time and *read*!

Problem P28. Please reproduce Table 7.1. That is, consider the scalar ODE IVP

$$u'(t) = \lambda\,(u(t) - \cos(t)) - \sin(t), \qquad u(0) = 1$$

Use $\lambda = -2100$. Apply forward Euler[1] to compute approximations of $u(T)$ for $T = 2$, for the given values of k, and report the final-time numerical errors $|U^N - u(T)|$ as in the Table. Confirm by this experiment that there is a critical value of k around 0.00095 where the error finally drops from enormous values to something comparable to, and then much smaller than, the solution magnitude itself.

The book's explains why: $|1+k\lambda| \le 1$ *only if* $k|\lambda| < 2$ *or equivalently* $k < 2/|\lambda| = 2/2100 = 0.00095238$. *There is no need to include this analysis in your answer; please just confirm it experimentally.*

Problem P29. Consider *θ-methods* for $u' = f(t,u)$:

$$U^{n+1} = U^n + k\Big[(1-\theta) f(t_n, U^n) + \theta f(t_{n+1}, U^{n+1})\Big]$$

Here $0 \le \theta \le 1$ is a fixed parameter.

(a) Cases $\theta = 0, 1/2, 1$ are all familiar methods. Name them. Then find the exact absolute stability regions for $\theta = 0, 1/4, 1/2, 3/4, 1$. (*Hint. Apply method to the test equation, and simplify. Write the complex number* $z = k\lambda$ *as* $z = x + iy$*. Find discs!*)

(b) Show they are A-stable for any $\theta \ge 1/2$.

Problem P30. **(a)** For the classical RK4 method, which is Example 5.13 in the textbook, show that the stability function is $R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4$. (*Hint. Apply to the test equation. I skipped details for this during lecture; please fill them in.*)

(b) Use a filled-contour plotter, like Matlab's `contourf` as shown in class, to plot the region of absolute stability of RK4. (*Hint. I did this in lecture for an RK2 method.*)

[1] Re-using an old code is just fine, but of course please pay attention to the details!

Problem P31. Consider this Runge-Kutta method, an *implicit* and one-step interpretation of the midpoint method:

$$U^* = U^n + \frac{k}{2} f\big(t_n + k/2, U^*\big),$$
$$U^{n+1} = U^n + k f\big(t_n + k/2, U^*\big).$$

The first stage uses backward Euler to (implicitly) compute a value at the midpoint. The second stage is a midpoint method using this value.[2] Please determine the region of absolute stability for this (combined) method; please do this exactly! Is this method A-stable? Is it L-stable?

Problem P32. For a famously stiff problem, consider the heat PDE

$$u_t = u_{xx} \tag{1}$$

Here $u(t, x)$ might be the temperature in a rod of length one ($0 \le x \le 1$). Let us set boundary temperatures to zero ($u(t, 0) = u(t, 1) = 0$), and assume some initial temperature distribution $u(0, x) = \eta(x)$.

Suppose we apply the *method of lines* (MOL) to (1). That is, we discretize the spatial derivatives using the notation from Chapter 2. Specifically, let us use $m + 1$ subintervals, $h = 1/(m + 1)$, and $x_j = jh$ for $j = 0, 1, 2, \dots, m + 1$. Now $U_j(t) \approx u(t, x_j)$. By eliminating unknowns $U_0 = 0$ and $U_{m+1} = 0$, and keeping the time derivatives as ordinary derivatives, from (1) we get a linear ODE system of dimension m,

$$U(t)' = A_m U(t) \tag{2}$$

where $U(t) \in \mathbb{R}^m$ and $A = A_m$ is *exactly* the matrix in the textbook's equation (2.10). Note that $U(0)_j = \eta(x_j)$ from the initial condition.

The eigenvalues of A_m are given by equation (2.23) in the textbook:

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right),$$

for $p = 1, \dots, m$.

(a) Please explain why all eigenvalues λ_p are points on the negative real axis. Then justify the following approximation of the largest-magnitude (and most negative) eigenvalue:

$$\lambda_m \approx -4(m + 1)^2.$$

(*Hint. Use a Taylor expansion of* $\cos(\theta)$ *around the right location.*)

(b) Suppose forward Euler is applied to solve (2) with equal time steps $k > 0$. How small must k be chosen so that all of the values $z_p = \lambda_p k$, for $p = 1, \dots, m$, are inside the region of absolute stability of forward Euler? Your answer will depend on m, but not p. You may use the approximation in part **(a)** for your analysis.

(c) What happens to the maximum stable time step for forward Euler, i.e. the answer from **b**, when you double the spatial grid resolution? With the same doubling, what happens to the cost of solving the heat equation problem out to some time $T > 0$?

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[2] One may show that this scheme has LTE $\tau^n = O(k^2)$, but here there is no request to do so.