Assignment #9

Due Wednesday, 26 April 2023, at the start of class

Please read textbook[1] Chapters 10 and 11.

Problem P37. Consider the following (*corrected*) method for solving the advection equation $u_t + au_x = 0$, where a is constant

$$U_i^{n+1} = U_i^{n-1} - \frac{ak}{h}(U_{i+1}^n - U_{i-1}^n).$$

This applies centered differences to all derivatives; it is the *leapfrog* method.

a) Determine the order of accuracy of the truncation error of this method. The answer will be in form $\tau(x,t) = O(k^p + h^q)$; determine p, q.

b) Apply a von Neumann analysis. (*Hint.* See some Worksheet solutions.)

c) State the MOL ODE system $U(t)' = AU(t)$ from which the above method comes. Assuming periodic boundary conditions on the interval $x \in [0,1]$, what are the eigenvalues of A? (*Hint.* Look them up in Chapter 10. In fact, you may extract most of this answer from the book. Give specific references and be brief!) Then derive the method by applying the midpoint ODE method to it. By looking up the stability region of the midpoint method, explain what is understood about the stability of this PDE method.

d) Implement this leapfrog method on the following *periodic boundary condition* problem: $x \in [0,1]$, $a = 0.5$, $t_f = 10$, $u(x,0) = \sin(6\pi x)$. To make the implementation work you will have to compute the first step by some other scheme; describe and justify what you do.

e) Noting that the final time is $t_f = 10$, the exact solution in part **d)** is $u(x,t_f) = \sin(6\pi x)$; explain why. Then use $h = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$ and $k = h$ and show a log-log convergence plot using the infinity norm for the error. What $O(h^p)$ do you expect for the rate of convergence, and what do you measure?

Problem P38. Consider the nonlinear Poisson equation in 2D

$$u_{xx} + u_{yy} + \gamma\, u^3 = g(x,y) \tag{1}$$

on the unit square $(x,y) \in [0,1] \times [0,1]$, subject to zero Dirichlet boundary conditions.

a) We can manufacture a solution because we are free to choose the right-hand side $g(x,y)$. Let us define

$$u(x,y) = \sin(\pi x)\sin(2\pi y). \tag{2}$$

This is a smooth function which satisfies the boundary conditions. Compute $g(x,y)$ so that (2) is an exact solution of (1); note that the $g(x,y)$ formula will depend on γ.

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

b) Based on the MATLAB program `heat2d.m`, for example, which is online at

bueler.github.io/nade/assets/codes/heat2d.m,

or a similar code for the 2D linear Poisson equation, write a numerical solver for (1). Use this approach:

- Use centered differences with spacing $h_x = h_y = 1/(m + 1)$.
- Approximate (1) by a nonlinear system

$$F(U) = 0 \tag{3}$$

  Here $U \in \mathbb{R}^N$ denotes the solution of the algebraic equations, $N = m^2$, and the function $F(V)$ is the *residual* for a current estimate V.
- Solve the algebraic equations (3) by Newton's method. Use $U^{[0]} = 0$ as the initial iterate.
- Stop the Newton iteration when the residual norm $\|F(U^{[k]})\|$ is reduced by 10^{-9} of the initial residual norm.
- Observe that when you set $\gamma = 0$ your code should solve the linear Poisson problem $u_{xx} + u_{yy} = g(x, y)$ by doing one Newton step. (This fact should not change how you write the code, but it may help in debugging.)

c) Show, using the exact solution from part **a)**, that with both $\gamma = 0$ (linear Poisson equation) and $\gamma = 10$ (a nonlinear case) your code exhibits the expected $O(h^2)$ convergence.