Assignment #8

Due Wednesday, 19 April 2023, at the start of class

Please read textbook[1] Chapters 8, 9, and 10. This assignment is mostly Chapter 9.

Problem P34. Consider the heat equation $u_t = D\,u_{xx}$ for $D > 0$ constant, $x \in [0,1]$, and Dirichlet boundary conditions $u(t,0) = 0$ and $u(t,1) = 0$. Suppose we have initial condition $u(0,x) = \sin(5\pi x)$.

a) Confirm that

$$u(t,x) = e^{-25\pi^2 Dt}\sin(5\pi x)$$

is an exact solution. (*It is* the *exact solution, but you do not have to show this.*)

b) Implement the backward Euler (BE) method, as applied to MOL ODE system (9.10), to solve this heat equation problem. Specifically, use diffusivity $D = 1/20$ and final time $t_f = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, a linear system, but you should use sparse storage and MATLAB's backslash or similar. Feel free to reuse or modify code from **P33** on Assignment #7.

c) Suppose we set $k = h$ for the "refinement path". (*Of course, for BE stability* does not *constrain our refinement path.*) What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from **a)**, at the final time, and the infinity norm $\|\cdot\|_\infty$, and $h = 0.02, 0.01, 0.005, 0.002, 0.001, 0.0005$. Make a log-log convergence plot of h versus the error.

d) Repeat parts **b)** and **c)** but with the trapezoidal rule, i.e. Crank-Nicolson (CN) equation (9.6). Use the same refinement path. Add the result to the same plot; turn in codes for BE and CN, but only one well-designed log-log convergence plot.

Problem P35. Consider the following scheme, which applies centered differences to both sides of the heat equation $u_t = u_{xx}$:

$$U_j^{n+2} = U_j^n + \frac{2k}{h^2}(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}).$$

This is called the *Richardson* method. (*L. F. Richardson did many things more important than, and successful than, inventing this scheme!*)

a) Compute the truncation error to determine the order of accuracy of this method, in space and time. The answer will be in form $\tau(t,x) = O(k^p + h^q)$; determine p, q.

b) Derive the method by applying the midpoint ODE method, equation (5.23), to the MOL ODE system (9.10). Also, find the region of absolute stability of the midpoint method (5.23); it is in the textbook. Is the method likely to generate reasonable results? Why or why not?

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

c) Do a von Neumann stability analysis of this scheme. What do you conclude?

Problem P36. Consider the Jacobi iteration[2] for the linear system $A\mathbf{u} = \mathbf{b}$ arising from a centered FD approximation of the boundary value problem $u''(x) = f(x)$. Here A is the matrix in equation (2.10); there is *no* need to rederive it. Show that this iteration can be interpreted as forward Euler time-stepping applied to a heat equation MOL system like (9.10), but *with time step* $k = \frac{1}{2}h^2$. Specifically, the MOL equations are those arising from a centered *spatial* FD discretization of $u_t(t,x) = u_{xx}(t,x) - f(x)$.

Comment 1. No implementations or coding is needed for this problem.

Comment 2. The solution of the time-dependent heat equation decays to the steady state solution, that is, to the solution of $u'' = f$. (This assumes steady boundary values.) However, while marching to steady state with an explicit method is one way to solve the steady-state boundary value problem, it is a very inefficient way. (Not recommended!) Instead, just focus on solving the system $A\mathbf{u} = \mathbf{b}$ quickly.

[2] The Jacobi iteration, equation (4.4) in the textbook, was covered in the "Classical iterative methods" slides bueler.github.io/nade/assets/slides/iterative.pdf