Assignment #7

Due Monday, 10 April 2023, at the start of class

Please read textbook[1] Chapters 6 and 7. Within this material we are de-emphasizing the discussion of multistep methods, so full understanding of sections 5.9, 6.4, 7.3, and 7.6.1 is not expected. Basically, full understanding of the other sections *is* expected. In any case, *actually reading* these Chapters is going to be important to success on this and later Assignments.

Problem P30. Consider the "θ-methods" for $u' = f(t,u)$, namely

$$U^{n+1} = U^n + k\Big[(1-\theta)f(t_n, U^n) + \theta f(t_{n+1}, U^{n+1})\Big],$$

where $0 \le \theta \le 1$ is a fixed parameter.[2]

a) Cases $\theta = 0, 1/2, 1$ are all familiar methods. Name them.

b) Find the (absolute) stability regions for $\theta = 0, 1/4, 1/2, 3/4, 1$. (*Hint.* Write the complex number $z = k\lambda$ as $z = x + iy$. Find the circles!)

c) Show that the θ-methods are A-stable for $\theta \ge 1/2$.

Problem P31. Consider this Runge-Kutta method, a one-step and implicit interpretation of the multistep midpoint method:

$$U^* = U^n + \frac{k}{2} f\big(t_n + k/2, U^*\big),$$
$$U^{n+1} = U^n + k f\big(t_n + k/2, U^*\big).$$

The first stage is backward Euler to determine an approximation to the value at the midpoint in time. The second stage is a midpoint method using this value.

a) Determine the order of accuracy of this method. That is, compute the truncation error accurately enough to know the power p in $\tau = O(k^p)$.

b) Determine the stability region. Is this method A-stable? Is it L-stable?

Problem P32. Reproduce Table 7.1. In particular, consider the scalar ODE IVP

$$u'(t) = \lambda\left(u(t) - \cos(t)\right) - \sin(t), \qquad u(0) = 1,$$

with the particular value $\lambda = -2100$. Use an implementation of forward Euler, for example from your or my solutions to **Assignment #6**, to compute approximations of $u(T)$ for $T = 2$, for the given values of k, and report the final-time numerical errors

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[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

[2] Note that I did all parts of this problem **P30** by hand.

$|U^N - u(T)|$ as in the Table. Confirm by this experiment[3] that there is a critical value of k around 0.00095 where the error finally drops from enormous values to something comparable to, then much smaller than, the solution magnitude itself.

Problem P33. For a famously stiff problem, consider the heat PDE

$$u_t = u_{xx} \tag{1}$$

Here $u(t,x)$ is the temperature in a rod of length one ($0 \le x \le 1$) and we set boundary temperatures to zero ($u(t,0) = 0$ and $u(t,1) = 0$). For an initial temperature distribution we set one part hotter than the rest:

$$u(0,x) = \begin{cases} 1, & 0.25 < x < 0.5, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose we seek $u(1,x)$, i.e. we set $t_f = 1$.

We apply the *method of lines* (MOL) to (1). That is, we discretize the spatial (x) derivatives using the notation from Chapter 2. Specifically, use $m+1$ subintervals, let $h = 1/(m+1)$, and let $x_j = jh$ for $j = 0, 1, 2, \dots, m+1$. Now $U_j(t) \approx u(t, x_j)$. By eliminating unknowns $U_0 = 0$ and $U_{m+1} = 0$, and keeping the time derivatives as ordinary derivatives, from (1) we get a linear ODE system of dimension m,

$$U(t)' = AU(t) \tag{2}$$

where $U(t) \in \mathbb{R}^m$ and A is *exactly* the matrix in the textbook's equation (2.10). For a given m, note $U(0)$ is computed from the above formula for $u(0,x)$.

a) Implement both forward and backward Euler on (2). For BE, store A using `sparse` storage and solve the equation using backslash or another linear solver which will automatically detect that the matrix is tridiagonal and solve it efficiently.

b) Now consider the $m = 99$ case, so $h = 0.01$, and let $k = t_f/N = 1/N$ be the time step length. For BE, compute and show the solution using $N = 100$ time steps. For FE, $N = 100$ will generate extraordinary explosion. (Confirm this but don't show it.) Determine the largest-possible absolutely-stable time step k from the eigenvalues of A and the stability region of FE. Finally, compare the computational costs of the two runs by counting floating-point multiplications.[4] You will conclude that an implicit is indeed effective in this case.

1 point of extra credit) Find the exact solution, presumably using a Fourier sine series. Plot it beside the $N = 100$ BE solution. BE looks pretty good on this problem!

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[3] Of course, the book explains the effect logically, which is the major point of Chapter 7, at least as it applies to forward Euler: $|1+k\lambda| \le 1$ only if $k|\lambda| < 2$ or equivalently $k < 2/|\lambda| = 2/2100 = 0.00095238$.

[4] For an $m \times m$ tridiagonal matrix A, Av costs $3m$ multiplications while $A^{-1}v$ costs $5m$ multiplications.