Assignment #5

Due Monday, 20 March 2023, at the start of class

Please read textbook[1] sections 5.1–5.8, plus Appendices C and D.

These problems are mostly about eigenvalues. I assume you have had an undergraduate course in linear algebra, and so this should be a review topic, but I start here with a quick summary. Appendices C and D cover more advanced eigen-topics.

By definition, a *nonzero* vector $v \in \mathbb{R}^m$ is an *eigenvector* of a square matrix $A \in \mathbb{R}^{m\times m}$ if it has the property that multiplication by A merely lengthens or shortens it:

$$Av = \lambda v. \tag{1}$$

The number λ is called the *eigenvalue* corresponding to v. (*"eigen" means something like "property of". That is, v and λ are in some sense "owned" by A.*) Now, if (1) is true then the matrix

$$\lambda I - A$$

has a vector in its *null space*. That is, there are nonzero vectors, at least v and nonzero multiples of v, that $\lambda I - A$ sends to zero:

$$(\lambda I - A)v = 0.$$

This equation is true, by a fundamental equivalence in linear algebra, if and only if $\lambda I - A$ is not invertible.

In particular, $\det(\lambda I - A) = 0$, which is a polynomial equation with real coefficients:

$$p(\lambda) = \det(\lambda I - A).$$

(*Recall we assumed A had real entries.*) Finding all the eigenvalues is equivalent to finding all the roots of this polynomial. Some of these roots may be complex:

$$\text{in general, } \lambda \in \mathbb{C}.$$

However, because the coefficients of the polynomial are real, if λ is not real then its conjugate $\bar\lambda$ is also a root of the polynomial and thus an eigenvalue of A.

Now suppose λ is an eigenvalue of A. Finding a corresponding eigenvector is the task of finding a vector in the null space of a matrix. In particular, the row operations of Gauss elimination will convert the equation $(\lambda I - A)v = 0$ into an upper triangular equation $Uv = 0$ where U is both upper triangular and has at least one row of zeros. (*This is because $\lambda I - A$ is not invertible.*) The matrix equation $Uv = 0$ can be used to generate every eigenvector corresponding to λ, the *eigenspace* for λ. This eigenspace has dimension at least one.

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

Problem P22. **a)** Compute *by hand* the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Show all your work. (*Hint: As you expand the determinant, watch for a factor to appear. You may* check *your work with* MATLAB.)

b) Continuing with the same matrix A, do the following using MATLAB etc., and show the command-line session or code: Choose a vector $u \in \mathbb{R}^3$ at random, for instance `u = randn(3,1)`. Apply A to it 50 times: $w = A^{50}u$. Now compute $\|Aw\|_2/\|w\|_2$. You will get the number 3.0000. Why? Explain in several sentences, using equations to make it clear.

Hint. If an $m \times m$ matrix has m distinct eigenvalues then the corresponding eigenvectors form a basis. Any vector can be written in this basis. On the other hand, multiplication by this A stretches one basis vector the most.

c) Note that $w = A^{50}u$ from part **(b)** is very large in norm. Why? For a random u, give an estimate of the norm of the vector $A^k u$ for large k.

Problem P23. *The purpose of this problem is to give you a visual sense of what can happen to the eigenvalues of non-symmetric real matrices.*

a) Consider this matrix-valued function of x:

$$M(x) = \begin{bmatrix} 2 & x & x \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Note that $M(-1) = A$ from **P22**. Also note that if $x \neq -1$ then $M(x)$ is not symmetric.

Use MATLAB to generate a single clear figure showing all the eigenvalues of all matrices $M(x)$ for $x \in [-1, 5]$. Label this figure in an attempt to clarify how the eigenvalues depend on x.

b) By doing a by-hand calculation, at what x value in the interval $[-1, 5]$ do non-real eigenvalues first appear? (*Hint.* Your answer will be compatible with your figure if you have done everything right.)

Problem P24. Check that the solution $u(t)$ given by Duhamel's principle, equation (5.8) in the textbook, satisfies ODE (5.6) and the initial condition $u(t_0) = \eta$.

Hint. Look up the Leibniz rule for differentiating an integral? To understand and explain the simple result of differentiating the matrix exponential, note you can differentiate the absolutely-convergent Taylor series (D.31), in Appendix D, term by term.

Regarding the next two Problems, recall A is diagonalizable if there is an *invertible* matrix R and a diagonal matrix Λ so that $AR = R\Lambda$ or equivalently $\Lambda = R^{-1}AR$. The diagonal entries in Λ are the eigenvalues of A and the columns of R are eigenvectors. Though R, Λ can be determined from MATLAB by the command `[R,Lambda] = eig(A)`, clarity in the next two Problems requires doing the eigenvalue calculations by hand. Feel free to check via MATLAB that you have the correct eigenvalues!

Problem P25. Consider the ODE system

$$\begin{aligned} u_1' &= 2u_1, \\ u_2' &= 3u_1 - 2u_2 \end{aligned}$$

with some initial conditions at $t = 0$: $u_1(0) = a, u_2(0) = b$.

Solve this system two ways:

a) Solve the first equation. Then insert this into the second equation to get a non-homogeneous linear ODE for u_2. Solve this using Duhamel's principle.

b) Write the system as $u' = Au$, compute the matrix exponential, and get the solution in the form of equation (D.30) in Appendix D.

Hint. The diagonalization of A in **(b)** can and should be done by hand. Simplify the results of each part enough to see they give the same solution.

Problem P26. The ODE IVP

$$v'' = -9v, \quad v(0) = v_0, \quad v'(0) = w_0$$

has solution

$$v(t) = v_0 \cos(3t) + \frac{w_0}{3}\sin(3t).$$

Verify this.

Construct this solution by first rewriting the ODE as a first-order system $u' = Au$. Then compute the solution $u(t) = e^{At}u(0)$ by using equation (D.30) in Appendix D.