Assignment #4

Due Wednesday, 1 March 2023, at the start of class (*revised*)

Please read sections 2.12–2.21 and 3.1–3.4 from the textbook.[1] Two of the problems below require you to remember that you can find the general solutions of constant-coefficient, linear, homogeneous ODEs *by hand* when needed.

Problem P18. (*A sometimes ill-posed boundary value problem.*)

a) Consider the following linear BVP with Dirichlet boundary conditions:

$$u''(x) + u(x) = 0 \quad \text{for } a < x < b, \qquad u(a) = \alpha, \qquad u(b) = \beta. \tag{1}$$

(*This equation arises from linearizing the pendulum equation (2.75), for example.*) Write a MATLAB/etc. finite difference code to solve this problem.

b) Determine the exact solution to the problem when $a = 0, b = 1, \alpha = 2, \beta = 3$. Test your code from part **a)** using this solution, including a demonstration of convergence at the optimal rate (*which is what?*) as $h \to 0$. Generate a convergence figure.

c) Let $a = 0$ and $b = \pi$. For what values of α and β does BVP (1) have solutions? Sketch a family of solutions in a case where there are infinitely-many solutions.

Problem P19. (*Nonlinear pendulum.*) Write a program to solve the BVP for the nonlinear pendulum as discussed in the text, i.e. problem (2.77), using the Newton iteration strategy outlined in subsection 2.16.1. Reproduce Figures 2.4(b) and 2.5, for which $T = 2\pi, \alpha = \beta = 0.7$.

Problem P20. Recall the ODE BVPs from **P11** on Assignment #2:

$$u''(x) + p(x)\,u'(x) + q(x)\,u(x) = f(x), \quad u(x_L) = \alpha, \quad u(x_R) = \beta.$$

a) Consider the case $x_L = 0, x_R = 1, \alpha = 1, \beta = 0, p = -20, q = 0$, and $f(x) = 0$. Confirm that an exact solution to this problem is

$$u(x) = 1 - \frac{1 - e^{20x}}{1 - e^{20}}.$$

Is it the only solution?

b) Solve the problem in part **a)** numerically using centered finite differences, $h = 1/(m+1)$ equal spacing, and $m = 3, 5, 10, 20, 50, 200, 1000$ interior points. Put all these numerical solutions, and the exact solution from **a)**, on one figure, with decent labeling. (Use `legend` or similar.)

[1] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Diff. Eqns.*, SIAM Press 2007

c) Observe that the solutions for small values of m are poor but the high m solutions all basically agree. Why do you think that small m values are problematic here, though they are not for the problem solved in section 2.4? Write a few sentences, perhaps based on a bit of research into section 2.17 or wikipedia pages. (*Hint.* Observe that the exact solution in part **a)** mostly does not "feel" one of the boundary conditions. Also consider the $p = +20$ case, with other data the same. What is the exact solution to the reduced equation $pu' = 0$ and what boundary conditions does it need?)

Problem P21. (*This problem is updated.*) (*Poisson equation on the unit square.*)

a) Based on the ideas in sections 3.1–3.3, namely centered finite differences, write a MATLAB/etc. code that solves Poisson equation (3.5) on the unit square $0 \le x \le 1, 0 \le y \le 1$, with zero Dirichlet boundary conditions. Allow the user to set the function $f = f(x, y)$. You may, as usual, use MATLAB's backslash, or equivalent, to solve the linear system.

Hint. Start from my code `heat2d.m` at
`https://bueler.github.io/nade/assets/codes/heat2d.m`.

b) Find an appropriate nonzero exact solution that allows you to verify that the code converges at the theoretically-expected rate. That is, show that $\|E^h\|_2 \to 0$, as $h \to 0$, at the rate $O(h^2)$ if $\Delta x = \Delta y = h = 1/(m+1)$.

c) Use MATLAB's `spy`, or similar, to show the sparsity pattern of the matrix A^h for $m = 5$. Also confirm in that case that the matrix has the form shown by equation (3.12).