Name: SOLUTIONS

Math 615 NADE (Bueler) Friday, 24 March 2023

Midterm Exam

In class. No textbook, notes, or internet. 70 minutes. 100 points possible.

1. (*10 pts*) Use Taylor's theorem to derive the centered finite difference (FD) approximation to $u''(x)$ for an equally-spaced grid. (*Hints.* Denote the grid spacing by h. Use Taylor twice. Combine and cancel terms.) To state your final result, fill in the blanks at the bottom.

$$u(x+h) = u(x) + u'(x)h + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + O(h^4)$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{3}u'''(x) + O(h^4)$$

add:

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + O(h^4)$$

rearrange:

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + O(h^2)$$

$$u''(x) = \frac{\boxed{u(x+h) - 2u(x) + u(x-h)}}{\boxed{h^2}} + O\left(h^{\boxed{2}}\right)$$

2. (*10 pts*) Consider the ODE BVP

$$(*) \qquad u''(x) + 5u(x) = f(x), \qquad u(0) = \alpha, \quad u(2) = \beta,$$

for $f(x)$ a given continuous function and α, β any real numbers. Propose an FD scheme, on an equally-spaced grid of m subintervals, for problem $(*)$. (*Hints.* Describe the grid. State the main FD equation. State/include how the boundary conditions are handled. Make the range of indices clear in each expression. You do not need to prove or explain anything.)

$$h = \frac{2-0}{m} = \frac{2}{m}, \qquad x_j = jh \quad [j = 0, 1, \ldots, m+1]$$

$$\frac{U^{j+1} - 2U^j + U^{j-1}}{h^2} + 5U^j = f(x_j) \quad [j = 1, \ldots, m-1]$$

$$U^0 = \alpha, \quad U^m = \beta$$

this is optional, if above is complete

becomes a linear system:

$$\begin{array}{cc} & m+1 \text{ cols} \\ \begin{array}{c} m+1 \\ \text{rows} \end{array} & \begin{bmatrix} \bullet & & & & & \\ \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & \bullet & & \\ & & & \ddots & & \\ & & & \bullet & \bullet & \bullet \\ & & & & & \bullet \end{bmatrix} \end{array} \begin{bmatrix} U^0 \\ U^1 \\ U^2 \\ \vdots \\ U^{m-1} \\ U^m \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{m-1}) \\ \beta \end{bmatrix}$$

$$A^h U^h = F^h$$

other possible approach is just fine:

$$U^h = \begin{bmatrix} U^1 \\ \vdots \\ U^{m-1} \end{bmatrix} \leftarrow \text{only interior points have unknowns}$$

b.c. in eqns

$$\frac{U^2 - 2U^1 + \alpha}{h^2} = f(x_1)$$

$$\vdots$$

3. **(a)** (*8 pts*) For an ODE problem and an FD scheme, for example as on the previous page, define the *local truncation error* τ^h. (*Hints.* You need only *define* it, not simplify or expand it.)

the local truncation error for problem 2 is

$$\tau^h = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + 5u(x_j) - f(x_j),$$

i.e. τ is residual from applying scheme to exact solution

(b) (*5 pts*) Define what it means for an FD scheme to be *consistent*.

an FD scheme is consistent if

$$\lim_{h \to 0} \tau^h = 0$$

(c) (*5 pts*) Suppose an FD scheme with grid spacing h is applied to a linear ODE or PDE BVP. This generates a matrix equation

$$A^h U^h = F^h.$$

Define what it means for the scheme to be *stable*.

an FD scheme which generates these linear systems is stable in norm $\|\cdot\|$ if there is a constant $C>0$ so that

$$\|(A^h)^{-1}\| \le C \quad \text{for all } h>0.$$

4. (*10 pts*) Consider a system of m nonlinear equations in m variables,

$$f_1(x_1, \dots, x_m) = 0,$$
$$\vdots$$
$$f_m(x_1, \dots, x_m) = 0.$$

and suppose each f_i is differentiable. We may write this system as "$F(x) = 0$" where $x \in \mathbb{R}^m$ and $F(x) \in \mathbb{R}^m$. State *Newton's method* for the equations $F(x) = 0$. (*Hints.* The method can be stated as a simple pseudocode or as a MATLAB function; syntax is not critical. It should start with an initial iterate $x^{[0]}$ and generate a sequence of iterates $x^{[k]}$, and have an appropriate termination criterion based on a tolerance parameter. *Required*: Make it clear which derivatives of $F(x)$ are used, and how, using standard notation as appropriate. *Required*: Clearly identify any linear systems that need to be solved.)

$x^{[0]}$

```
function z = newton(F, J, x0, tol)
    x = x0
    for j = 1:100   <- or "maxiter" etc.
        if ||F(x)|| < tol
            break
        end
        solve J(x) s = -F(x) for s
        x = x + s
    end
    z = x
```

$x^{[k+1]}$ $x^{[k]}$

linear system; solve e.g. by: $s = -J(x) \backslash F(x)$

here F is a function $F : \mathbb{R}^m \to \mathbb{R}^m$ where

$F(x)_j = f_j(x_1, \dots, x_m)$ and J is a function

$J : \mathbb{R}^m \to \mathbb{R}^{m \times m}$, the Jacobian, where

$$J(x)_{ij} = \frac{\partial f_i}{\partial x_j}(x_1, \dots, x_m)$$

5. **(a)** (*5 pts*) Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -8 & 0 \end{bmatrix}.$$

$$p(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-3 & & \\ & \lambda-2 & 1 \\ & 8 & \lambda \end{bmatrix} \right)$$

$$= (\lambda-3)\left[(\lambda-2)\lambda - 8\right] = (\lambda-3)(\lambda^2-2\lambda-8)$$

$$= (\lambda-3)(\lambda+2)(\lambda-4) \quad \therefore \quad \lambda_1=-2,\ \lambda_2=3,\ \lambda_3=4$$

are eigenvalues of A

(b) (*5 pts*) If λ_i is an eigenvalue of A, describe in a couple of sentences how to find a corresponding eigenvector. (*Hint.* Use complete sentences. *Required*: State the equation satisfied by the eigenvector.)

Given A and eigenvalue λ_i of A, we must find nonzero solutions of the linear system

$$Av = \lambda_i v$$

equivalently $(\lambda_i I - A)v = 0$. Nonzero solutions exist because $\lambda_i I - A$ is not invertible.

(c) (*5 pts*) For the matrix A in part **(a)**, will the iteration

$$y_{k+1} = Ay_k$$

converge for any nonzero vector $y_0 \in \mathbb{R}^3$? Explain, using precise language as appropriate.

No. Our convergence theorem for $y_{k+1} = My_k + c$ says this converges for all y_0 if and only if $\rho(M) < 1$. However, $\rho(A) = \max\{|\lambda_i|\} = 4 > 1$.

also o.k.: expand y_0 in eigenvectors and show divergence

6. (*10 pts*) For the ODE system

$$u'(t) = f(t, u(t)),$$

where $u(t) \in \mathbb{R}^s$, and supposing an initial condition $u(t_0) = \eta$ for $\eta \in \mathbb{R}^s$, write a pseudocode for the *explicit midpoint (EM) method*. The user provides a final time t_f and a fixed number N of steps. (*Hint.* State the method as a pseudocode or MATLAB function; syntax is not critical.)

scheme:

$$\left[\begin{aligned} U^* &= U^n + \tfrac{k}{2} f(t_n, U^n) \\ U^{n+1} &= U^n + k f(t_n + \tfrac{k}{2}, U^*) \end{aligned}\right]$$

```
function U = em(f, eta, t0, tf, N)
  k=(tf-t0)/N;   t = t0:k:tf;
  U=eta;
  for j=1:N
      Ustar = U+ (k/2) * f(t(j-1), U);
      U=U+ k * f(t(j-1)+k/2, Ustar);
  end
```

7. (*10 pts*) For the scalar ODE IVP

$$u'(t) = t - 5u(t), \quad u(0) = 3,$$

compute U^1 from one step of the *backward Euler* method, assuming $k = 0.5$.

$$\left.\frac{U^{n+1} - U^n}{k} = t_{n+1} - 5U^{n+1}\right] \text{scheme}$$

$$\left.\begin{matrix} n=0,\ k=0.5,\ t_0=0, \\ t_1=0.5,\ U^0=3 \end{matrix}\right\} \Rightarrow 2(U^1 - 3) = 0.5 - 5U^1$$

$$2U^1 + 5U^1 = 0.5 + 6$$

$$U^1 = \frac{6.5}{7} = \frac{13}{14}$$

8. (*7 pts*) For a square matrix $A \in \mathbb{R}^{m\times m}$, define the *matrix exponential* e^A.

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

absolutely convergent Taylor series of e^z

also o.k. if it is made clear that some matrices are <u>not</u> diagonalizable:

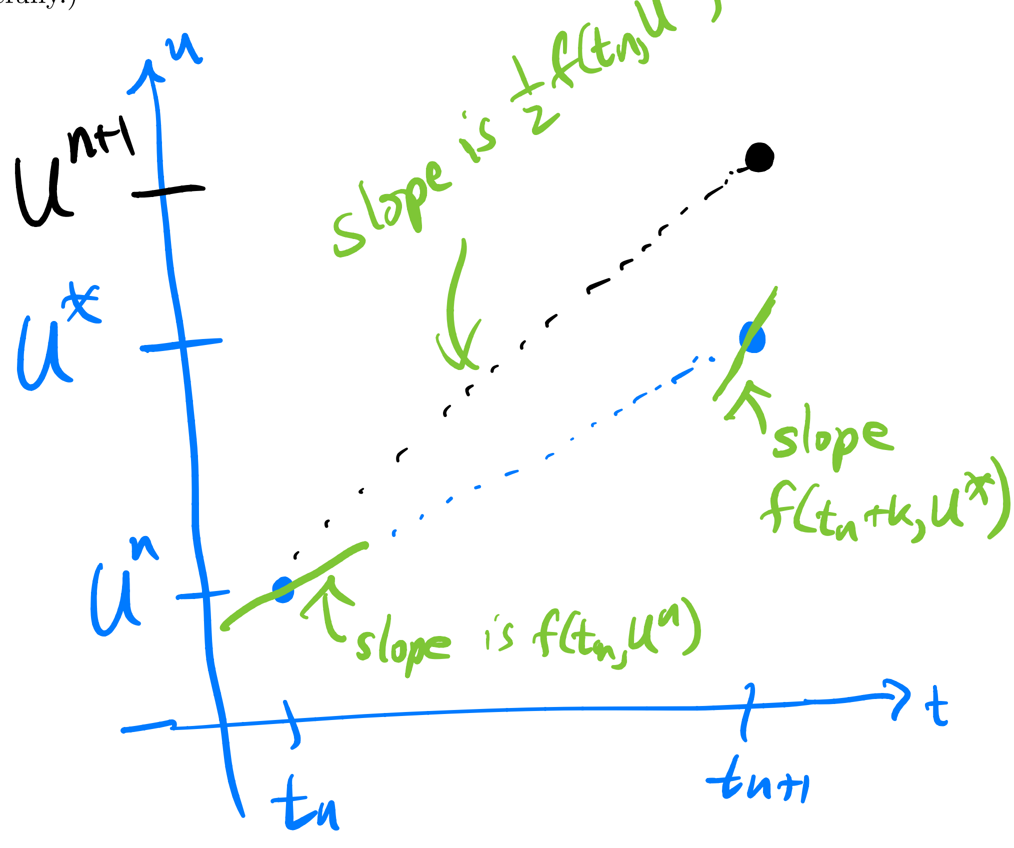
if $\underbrace{A = X \Lambda X^{-1}}_{\text{not always possible}}$ then $e^A = X e^{\Lambda} X^{-1}$

$$= X \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_m} \end{bmatrix} X^{-1}$$

9. (*10 pts*) Sketch, in the (t, u) plane, and showing slopes for the locations where $f(t, u(t))$ is evaluated, a picture/cartoon of a single step of the *explicit trapezoid (ET) method*:

$$U^* = U^n + k f(t_n, U^n)$$
$$U^{n+1} = U^n + \frac{k}{2}\left(f(t_n, U^n) + f(t_n + k, U^*)\right)$$

(*Hint.* Annotate your sketch carefully.)



recall: $\tau^h = A^h \hat{U}^h - F^h$

Extra Credit. (*3 pts*) Recalling the definitions and notation from problem **3**, prove the fundamental theorem of finite difference methods, essentially the Lax equivalence theorem, as it applies to ODE BVPs:

$$O(h^p) \text{ local truncation error} \quad + \quad \text{stability} \quad \Longrightarrow \quad O(h^p) \text{ numerical error.}$$

given: $\tau^h = O(h^p)$, $A^h U^h = F^h$, $\hat{U}^h$ is exact soln.,

and $\exists C$ s.t. $\|(A^h)^{-1}\| \le C$.

let: $E^h = U^h - \hat{U}^h$. Then

$$A^h E^h = A^h U^h - A^h \hat{U}^h$$
$$= F^h - (\tau^h + F^h) = -\tau^h$$

so: $\|E^h\| = \|(A^h)^{-1}\tau^h\| \le C\|\tau^h\| = C\,O(h^p)$
$= O(h^p)$

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