Name:

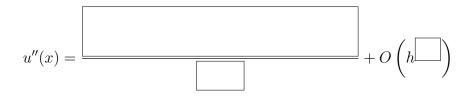
Math 615 NADE (Bueler)

Friday, 24 March 2023

## Midterm Exam

## In class. No textbook, notes, or internet. 70 minutes. 100 points possible.

1. (10 pts) Use Taylor's theorem to derive the centered finite difference (FD) approximation to u''(x) for an equally-spaced grid. (*Hints.* Denote the grid spacing by h. Use Taylor twice. Combine and cancel terms.) To state your final result, fill in the blanks at the bottom.



**2.**  $(10 \ pts)$  Consider the ODE BVP

(\*) 
$$u''(x) + 5u(x) = f(x), \quad u(0) = \alpha, \quad u(2) = \beta$$

for f(x) a given continuous function and  $\alpha, \beta$  any real numbers. Propose an FD scheme, on an equally-spaced grid of *m* subintervals, for problem (\*). (*Hints.* Describe the grid. State the main FD equation. State/include how the boundary conditions are handled. Make the range of indices clear in each expression. You do not need to prove or explain anything.)

3. (a) (8 pts) For an ODE problem and an FD scheme, for example as on the previous page, define the *local truncation error*  $\tau^h$ . (*Hints.* You need only *define* it, not simplify or expand it.)

(b) (5 pts) Define what it means for an FD scheme to be consistent.

(c) (5 pts) Suppose an FD scheme with grid spacing h is applied to a linear ODE or PDE BVP. This generates a matrix equation

$$A^h U^h = F^h.$$

Define what it means for the scheme to be *stable*.

## 4. (10 pts) Consider a system of m nonlinear equations in m variables,

$$f_1(x_1, \dots, x_m) = 0,$$
  
$$\vdots$$
  
$$f_m(x_1, \dots, x_m) = 0.$$

and suppose each  $f_i$  is differentiable. We may write this system as "F(x) = 0" where  $x \in \mathbb{R}^m$  and  $F(x) \in \mathbb{R}^m$ . State Newton's method for the equations F(x) = 0. (Hints. The method can be stated as a simple pseudocode or as a MATLAB function; syntax is not critical. It should start with an initial iterate  $x^{[0]}$  and generate a sequence of iterates  $x^{[k]}$ , and have an appropriate termination criterion based on a tolerance parameter. Required: Make it clear which derivatives of F(x) are used, and how, using standard notation as appropriate. Required: Clearly identify any linear systems that need to be solved.)

5. (a) (5 pts) Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -8 & 0 \end{bmatrix}.$$

(b) (5 pts) If  $\lambda_i$  is an eigenvalue of A, describe in a couple of sentences how to find a corresponding eigenvector. (*Hint.* Use complete sentences. *Required*: State the equation satisfied by the eigenvector.)

(c) (5 pts) For the matrix A in part (a), will the iteration

$$y_{k+1} = Ay_k$$

converge for any nonzero vector  $y_0 \in \mathbb{R}^3$ ? Explain, using precise language as appropriate.

## **6.** (10 pts) For the ODE system

$$u'(t) = f(t, u(t)),$$

where  $u(t) \in \mathbb{R}^s$ , and supposing an initial condition  $u(t_0) = \eta$  for  $\eta \in \mathbb{R}^s$ , write a pseudocode for the *explicit midpoint (EM) method*. The user provides a final time  $t_f$  and a fixed number N of steps. (*Hint*. State the method as a pseudocode or MATLAB function; syntax is not critical.)

**7.**  $(10 \ pts)$  For the scalar ODE IVP

$$u'(t) = t - 5u(t), \quad u(0) = 3,$$

compute  $U^1$  from one step of the backward Euler method, assuming k = 0.5.

8. (7 *pts*) For a square matrix  $A \in \mathbb{R}^{m \times m}$ , define the *matrix exponential*  $e^A$ .

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**9.** (10 pts) Sketch, in the (t, u) plane, and showing slopes for the locations where f(t, u(t)) is evaluated, a picture/cartoon of a single step of the *explicit trapezoid* (ET) method:

$$\begin{split} U^* &= U^n + k f(t_n, U^n) \\ U^{n+1} &= U^n + \frac{k}{2} \left( f(t_n, U^n) + f(t_n + k, U^*) \right) \end{split}$$

(*Hint.* Annotate your sketch carefully.)

**Extra Credit.** (3 pts) Recalling the definitions and notation from problem **3**, prove the fundamental theorem of finite difference methods, essentially the Lax equivalence theorem, as it applies to ODE BVPs:

 $O(h^p)$  local truncation error + stability  $\implies O(h^p)$  numerical error.

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