Name: ______________________________

Math 615 NADE (Bueler) Friday, 24 March 2023

Midterm Exam

In class. No textbook, notes, or internet. 70 minutes. 100 points possible.

1. (*10 pts*) Use Taylor's theorem to derive the centered finite difference (FD) approximation to $u''(x)$ for an equally-spaced grid. (*Hints.* Denote the grid spacing by h. Use Taylor twice. Combine and cancel terms.) To state your final result, fill in the blanks at the bottom.

$$u''(x) = \frac{\boxed{}}{\boxed{}} + O\left(h^{\boxed{}}\right)$$

2. (*10 pts*) Consider the ODE BVP

$$(*) \qquad u''(x) + 5u(x) = f(x), \qquad u(0) = \alpha, \quad u(2) = \beta,$$

for $f(x)$ a given continuous function and α, β any real numbers. Propose an FD scheme, on an equally-spaced grid of m subintervals, for problem $(*)$. (*Hints.* Describe the grid. State the main FD equation. State/include how the boundary conditions are handled. Make the range of indices clear in each expression. You do not need to prove or explain anything.)

3. **(a)** (*8 pts*) For an ODE problem and an FD scheme, for example as on the previous page, define the *local truncation error* τ^h. (*Hints.* You need only *define* it, not simplify or expand it.)

(b) (*5 pts*) Define what it means for an FD scheme to be *consistent.*

(c) (*5 pts*) Suppose an FD scheme with grid spacing h is applied to a linear ODE or PDE BVP. This generates a matrix equation

$$A^h U^h = F^h.$$

Define what it means for the scheme to be *stable.*

4. (*10 pts*) Consider a system of m nonlinear equations in m variables,

$$f_1(x_1, \dots, x_m) = 0,$$
$$\vdots$$
$$f_m(x_1, \dots, x_m) = 0.$$

and suppose each f_i is differentiable. We may write this system as "$F(x) = 0$" where $x \in \mathbb{R}^m$ and $F(x) \in \mathbb{R}^m$. State *Newton's method* for the equations $F(x) = 0$. (*Hints.* The method can be stated as a simple pseudocode or as a MATLAB function; syntax is not critical. It should start with an initial iterate $x^{[0]}$ and generate a sequence of iterates $x^{[k]}$, and have an appropriate termination criterion based on a tolerance parameter. *Required*: Make it clear which derivatives of $F(x)$ are used, and how, using standard notation as appropriate. *Required*: Clearly identify any linear systems that need to be solved.)

5. (a) (*5 pts*) Compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -8 & 0 \end{bmatrix}.$$

(b) (*5 pts*) If λ_i is an eigenvalue of A, describe in a couple of sentences how to find a corresponding eigenvector. (*Hint.* Use complete sentences. *Required*: State the equation satisfied by the eigenvector.)

(c) (*5 pts*) For the matrix A in part **(a)**, will the iteration

$$y_{k+1} = Ay_k$$

converge for any nonzero vector $y_0 \in \mathbb{R}^3$? Explain, using precise language as appropriate.

6. (*10 pts*) For the ODE system

$$u'(t) = f(t, u(t)),$$

where $u(t) \in \mathbb{R}^s$, and supposing an initial condition $u(t_0) = \eta$ for $\eta \in \mathbb{R}^s$, write a pseudocode for the *explicit midpoint (EM) method.* The user provides a final time t_f and a fixed number N of steps. (*Hint.* State the method as a pseudocode or MATLAB function; syntax is not critical.)

7. (*10 pts*) For the scalar ODE IVP

$$u'(t) = t - 5u(t), \quad u(0) = 3,$$

compute U^1 from one step of the *backward Euler* method, assuming $k = 0.5$.

8. (*7 pts*) For a square matrix $A \in \mathbb{R}^{m \times m}$, define the *matrix exponential* e^A.

9. (*10 pts*) Sketch, in the (t, u) plane, and showing slopes for the locations where $f(t, u(t))$ is evaluated, a picture/cartoon of a single step of the *explicit trapezoid (ET) method*:

$$U^* = U^n + kf(t_n, U^n)$$

$$U^{n+1} = U^n + \frac{k}{2}\left(f(t_n, U^n) + f(t_n + k, U^*)\right)$$

(*Hint.* Annotate your sketch carefully.)

Extra Credit. (*3 pts*) Recalling the definitions and notation from problem **3**, prove the fundamental theorem of finite difference methods, essentially the Lax equivalence theorem, as it applies to ODE BVPs:

$$O(h^p) \text{ local truncation error} \quad + \quad \text{stability} \quad \Longrightarrow \quad O(h^p) \text{ numerical error.}$$

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