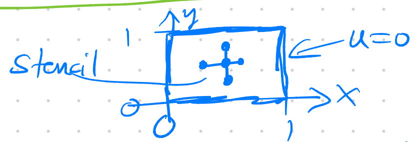
BUELER

This is example *pretends* that the 2D heat equation (9.29) is an allowed problem ... in fact it is not.

problem PDE (d)



The PDE IBVP is

$$u_t = u_{xx} + u_{yy}$$

on $(t,x,y) \in [0,t_f] \times [0,1] \times [0,1]$ with Dirichlet boundary values as shown. Centered FD gives MOL system

$$U_{jk}'(t) = \frac{U_{j+1,k} - 2U_{jk} + U_{j-1,k}}{h_x^2} + \frac{U_{j,k+1} - 2U_{jk} + U_{j,k-1}}{h_y^2}.$$

We consider forward Euler and backward Euler time-stepping, thus $\tau = O(\Delta t^1 + h_x^2 + h_y^2)$ in both cases. Note FE $\Leftrightarrow$ FTCS, BE $\Leftrightarrow$ BTCS as fully-discrete schemes. FTCS is conditionally stable with $\Delta t \le O(h_x^2 + h_y^2)$, but BTCS is unconditionally stable because BE is A-stable. Note FTCS is easy to program, but each time step of BTCS requires solving a linear system $A^h U = F^h$ where F^h includes the previous time step.

Here A^h is a symmetric matrix, and not tridiagonal, and sparse linear algebra is appropriate. The eigenvalues of A^h are negative so A-stable time stepping makes sense.

A suitable exact solution for testing is

$$u(t,x,y) = e^{-\pi^2(p^2+q^2)t} \sin(p\pi x)\sin(q\pi y).$$