Worksheet: elimination, via row operations and matrices

Do these calculations with a group, if possible.

A. Consider this system of three equations in three unknowns:

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ -8x_1 + 4x_2 \quad\quad &= -8 \\ 5x_1 + 7x_2 - 4x_3 &= -7 \end{aligned}$$

Do elimination, as described in lecture and in section 2.2, followed by back-substitution, to solve this system. Make sure to follow the standard ordering of operations, even when it is tempting to do some *ad hoc* steps! Show your work in a reasonable way; documenting each step here will help in problem **B** on the other side. (*Hint. The entries of* $\mathbf{x}$ *are small integers.*)

stage 1

$x_1 + x_2 + x_3 = 4$

$R_2 \leftarrow R_2 - \left(\frac{-8}{1}\right) R_1$: $\quad 12x_2 + 8x_3 = 24$

$R_3 \leftarrow R_3 - \left(\frac{5}{1}\right) R_1$: $\quad 2x_2 - 9x_3 = -27$

stage 2

$x_1 + x_2 + x_3 = 4$

$12x_2 + 8x_3 = 24$

$R_3 \leftarrow R_3 - \left(\frac{2}{12}\right) R_2$: $\quad -\frac{31}{3} x_3 = -31$

$-27 - \frac{1}{6} 24$

$= -27 - 4$

back-substitution

$x_3 = \frac{-31}{-31/3} = 3$

$x_2 = \frac{24 - 8(3)}{12} = 0$

$x_1 = \frac{4 - 0 - 3}{1} = 1$

B. Write the system in problem **A** (previous page) using matrices and vectors, as $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -8 & 4 & 0 \\ 5 & 7 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ -8 \\ -7 \end{bmatrix}$$

Next, write down the three elimination matrices that correspond to the row operations you did. That is, fill-in these 3×3 matrices:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix}$$

Finally, multiply-out matrices, as follows. You should see that all the elimination steps on the previous page are reproduced here.

stage 1

$$E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 12 & 8 \\ 5 & 7 & -4 \end{bmatrix} \qquad E_{21}\mathbf{b} = \begin{bmatrix} 4 \\ 24 \\ -7 \end{bmatrix}$$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 12 & 8 \\ 0 & 2 & -9 \end{bmatrix} \qquad E_{31}(E_{21}\mathbf{b}) = \begin{bmatrix} 4 \\ 24 \\ -27 \end{bmatrix}$$

stage 2

$$E_{32}(E_{31}(E_{21}A)) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 12 & 8 \\ 0 & 0 & -31/3 \end{bmatrix} \qquad E_{32}(E_{31}(E_{21}\mathbf{b})) = \begin{bmatrix} 4 \\ 24 \\ -31 \end{bmatrix}$$

C. Thus elimination converts the linear system to an upper triangular system:

$$A\mathbf{x} = \mathbf{b} \qquad \longrightarrow \qquad U\mathbf{x} = \mathbf{c}$$

What are U and $\mathbf{c}$?:

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 12 & 8 \\ 0 & 0 & -31/3 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 4 \\ 24 \\ -31 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Do back-substitution to (again) solve the system and find $\mathbf{x}$.

$$x_3 = \frac{-31}{-31/3} = 3, \qquad x_2 = \frac{24 - 8 \cdot 3}{12} = 0,$$

$$x_1 = \frac{4 - 0 - 3}{1} = 1$$