Worksheet: 4 curve-fitting problems

The four graphs below show solutions of data-fitting problems. The curve goes exactly through the data in problem **1**. It comes as close to the data as possible, in the least-squares sense, in problems **2**, **3**, **4**. We think of this as solving a linear system, but in fact when the number of points exceeds the number of parameters the system is over-determined. The system is "solved" via the normal equations as in §4.3:

$$\text{“}A\mathbf{v} = \mathbf{b}\text{”} \qquad \overset{\text{replaced by}}{\to} \qquad A^\top A\mathbf{v} = A^\top \mathbf{b}.$$

Problem **1** solves a square linear system as usual, and there is nothing to do.

For problems **2**, **3**, **4**, I have shown A and $\mathbf{b}$ and plotted the data. Using Matlab, you should

i) Confirm A and $\mathbf{b}$ for the given data and the given form of $p(x)$.
ii) Input A and $\mathbf{b}$, and form $A^\top A$ and $A^\top \mathbf{b}$.
iii) Solve the normal equations $A^\top A\mathbf{v} = A^\top \mathbf{b}$ to get $\mathbf{v}$ for the curve I plotted.
iv) Examine the projection $P = A(A^\top A)^{-1}A^\top$ and the vector $P\mathbf{b} = A\mathbf{v}$.

1. quadratic exactly fits 3 points

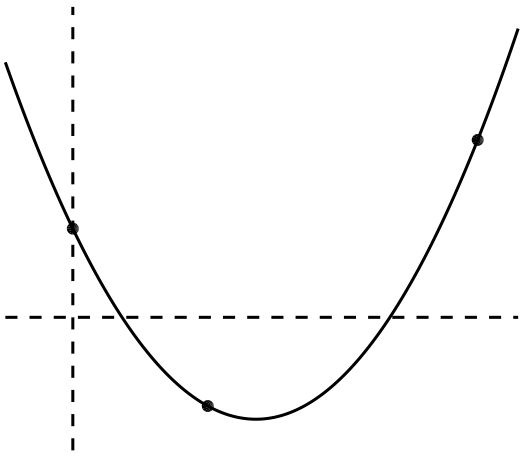
data (x, y): $(0, 1)$, $(1, -1)$, $(3, 2)$

curve: $p(x) = v_1 + v_2 x + v_3 x^2$

solved: $A\mathbf{v} = \mathbf{b}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\implies \quad \mathbf{v} = \begin{bmatrix} 1 \\ -19/6 \\ 7/6 \end{bmatrix}$$



2. line least-squares fits 4 points

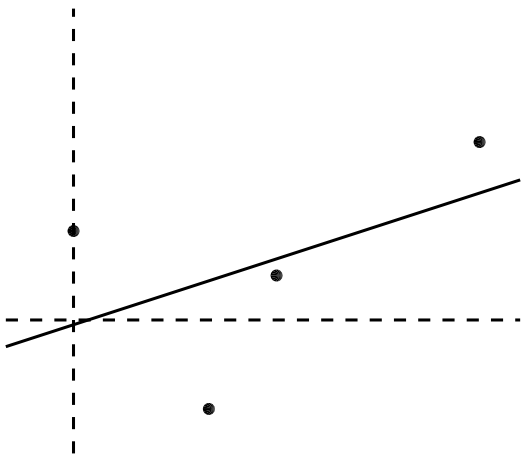
data (x, y): $(0, 1)$, $(1, -1)$, $(1.5, 0.5)$, $(3, 2)$

curve: $p(x) = v_1 + v_2 x$

solved: $A^\top A\mathbf{v} = A^\top \mathbf{b}$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1.5 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$\implies \quad \mathbf{v} = \begin{bmatrix} -4/75 \\ 37/75 \end{bmatrix}$$



3. quadratic least-squares fits 6 points

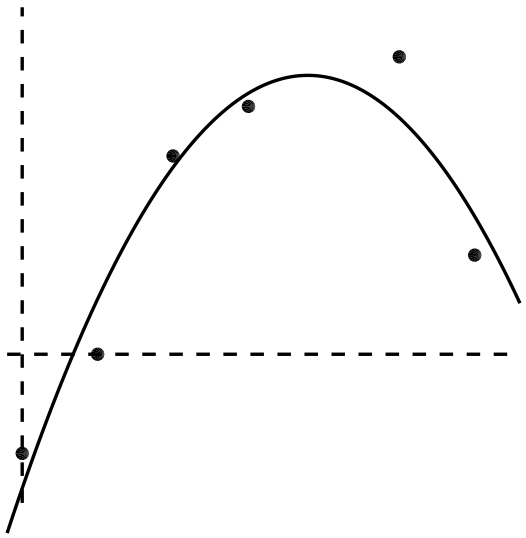
data (x, y): $(0, -1)$, $(0.5, 0)$, $(1, 2)$, $(1.5, 2.5)$, $(2.5, 3)$, $(3, 1)$

curve: $p(x) = v_1 + v_2 x + v_2 x^2$

solved: $A^\top A\mathbf{v} = A^\top \mathbf{b}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \\ 1 & 1.5 & 2.25 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2.5 \\ 3 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \quad \mathbf{v} = \begin{bmatrix} -1.35238 \\ 4.40000 \\ -1.16190 \end{bmatrix}$$



4. trigonometric least-squares fits 6 points

data (x, y): $(0, -1)$, $(0.5, 0)$, $(1, 2)$, $(1.5, 2.5)$, $(2.5, 3)$, $(3, 1)$

curve: $p(x) = v_1 + v_2 \sin x + v_2 \cos x$

solved: $A^\top A\mathbf{v} = A^\top \mathbf{b}$

$$A = \begin{bmatrix} 1 & \sin(0) & \cos(0) \\ 1 & \sin(0.5) & \cos(0.5) \\ 1 & \sin(1) & \cos(1) \\ 1 & \sin(1.5) & \cos(1.5) \\ 1 & \sin(2.5) & \cos(2.5) \\ 1 & \sin(3) & \cos(3) \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2.5 \\ 3 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \quad \mathbf{v} = \begin{bmatrix} -0.15410 \\ 3.00287 \\ -1.08695 \end{bmatrix}$$