SOLUTIONS

Worksheet: eigenstuff, and diagonalizing matrices

(a) For each matrix, compute the eigenvalues and eigenvectors by hand. Confirm your result using computer assistance.

(b) Is the matrix diagonalizable? If it is, form a (convenient) invertible matrix X of eigenvectors, and a diagonal matrix Λ of eigenvalues

(c) If the matrix was diagonalizable, confirm that $A = X\Lambda X^{-1}$. This step may be done with computer assistance.

1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\lambda_1 = 1$: $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

or any nonzero multiple!

$\lambda_2 = 2$: $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 3$: $\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdots \Rightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ are linearly-independent, $X = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible. So A is diagonalizable

$$A = X \Lambda X^{-1} \quad \text{with} \quad \Lambda = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

Matlab:
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>> A = ...
>> [X,D]=eig(A)      <- check X, Λ
>> X*D*inv(X)        <- check equals A
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2. $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad p(\lambda) = (1-\lambda)(1-\lambda) = 0 \quad \therefore \quad \lambda_1 = 1,\ \lambda_2 = 1$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\underline{\lambda_1 = 1:}$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

but $\vec{x}_2 = c\,\vec{x}_1$! so B is <u>not</u> diagonalizable

(one-dimensional space of eigenvectors)

3. $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad p(\lambda) = (1-\lambda)(1-\lambda)^2 - (1-\lambda) = (1-\lambda)((1-\lambda)^2 - 1)$

$$= (1-\lambda)(\lambda^2 - 2\lambda) = \lambda(1-\lambda)(\lambda-2) = 0$$

$\underline{\lambda_1 = 0:}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Rightarrow \vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = 1:}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Rightarrow \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda_3 = 2:}$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Rightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

so C is diagonalizable:

$$C = X \Lambda X^{-1} \quad \text{with} \quad \Lambda = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

(Matlab check as for A.)