SOLUTIONS

Worksheet: From linear system to LU factorization

Do these problems with a group, if possible!

I. Consider the following linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_1 + x_2 - 9x_3 &= -6 \\ 4x_1 - 3x_2 - 21x_3 &= -20 \\ -6x_1 - 13x_2 + 24x_3 &= 5 \end{aligned}$$

Solve the linear system by *elimination* and then *back-substitution*. Use the standard algorithm. Show your work, and in particular show, as an intermediate stage, the triangular system which you get after elimination.

$$\left[\begin{array}{ccc|c} 2 & 1 & -9 & -6 \\ 4 & -3 & -21 & -20 \\ -6 & -13 & 24 & 5 \end{array}\right] \qquad = A$$

elimination

$\ell_{21} = 2$ $\quad \ell_{31} = -3$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + 3R_1$

$$\left[\begin{array}{cccc} 2 & 1 & -9 & -6 \\ 0 & -5 & -3 & -8 \\ 0 & -10 & -3 & -13 \end{array}\right]$$

$= \vec{c}$

The Pivots

$\ell_{32} = 2$

$R_3 \leftarrow R_3 - 2R_2$

$$\left[\begin{array}{cccc} 2 & 1 & -9 & -6 \\ 0 & -5 & -3 & -8 \\ 0 & 0 & 3 & 3 \end{array}\right] \leftarrow \text{triangular system } U\vec{x} = \vec{c}$$

$= U$

back-subst.

$3x_3 = 3 \quad \to \quad x_3 = 1$

$-5x_2 - 3x_3 = -8 \quad \to x_2 = \dfrac{-8 + 3x_3}{-5} = 1$

$2x_1 + x_2 - 9x_3 = -6 \to x_3 = \dfrac{-6 - x_2 + 9x_3}{2} = 1$

note: this is solving $U\vec{x} = \vec{b}$

$$\to \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

II. **(a)** From problem **I**, what elimination matrices $E_{21}, E_{31}E_{32}$ did the row operations?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(b) What are the inverses $L_{21} = E_{21}^{-1}$, $L_{31} = E_{31}^{-1}$, $L_{32} = E_{32}^{-1}$?

$$L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad L_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(c) From problem **I**, what numbers were the pivots? What is the determinant of A?

pivots: $2, -5, 3$ $\rightarrow$ $\det(A) = 2 \cdot (-5) \cdot 3$

$= -30$

(d) The computation in problem **I** can regarded as factoring $A = LU$. What lower triangular matrix L and upper triangular matrix U were computed?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -9 \\ 0 & -5 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

(e) Multiply LU and confirm you get the original matrix A.

$$LU = \begin{bmatrix} 2 & 1 & -9 \\ 4 & -3 & -21 \\ -6 & -13 & 24 \end{bmatrix} = A \checkmark$$