Math 314
Exam 1

Name : Solutions
October 20, 2014

Show your work on all problems.

1. (19 pts.) A matrix A has LU factorization given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following questions about A. (You should *not* need to multiple L and U to get A).

(a) (3 pts.) Describe exactly the 3rd elimination step that is performed on A to reduce it to echelon form.

From L, the $(3,2)$ entry of 4 shows row 2 is multiplied by 4 and subtracted from row 3.

(b) (4 pts.) Are there vectors **b** for which $A\mathbf{x} = \mathbf{b}$ has no solution? Explain your reasoning.

Yes. Since U has 2 pivots, $\operatorname{rank}(A) = 2$, so $\mathcal{C}(A)$ is 2-dimensional in $\mathbb{R}^3$. Thus there are vectors $\vec{b}$ not in the column space, and for those $A\vec{x} = \vec{b}$ is not solvable

Alternate Solution: Yes. Since U has a row of 0's for some $\vec{b}$ elimination will lead to the last equation being $0=1$, so there is no solution.

(c) (7 pts.) Give a basis for the nullspace of A.

$$\left.\begin{aligned} 2x - y + z + w &= 0 \\ y + 2z + w &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} y &= -2z - w \\ x &= \tfrac{1}{2}(y - z - w) = \tfrac{1}{2}(-2z - w - z - w) = -\tfrac{3}{2}z - w \end{aligned}$$

z, w free

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}z - w \\ -2z - w \\ z \\ w \end{pmatrix} = z\begin{pmatrix} -\frac{3}{2} \\ -2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

basis is $\left(-\frac{3}{2}, -2, 1, 0\right), (-1, -1, 0, 1)$

(d) (5 pts.) For $\mathbf{b} = (1, 3, 1)$, a solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = (1, 1, 0, 0)$. Could this be the only solution? If not, give all solutions.

No, Solutions would be $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -\frac{3}{2} \\ -2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad z, w \in \mathbb{R}$

2. (12 pts. – 6 pts. each) If A is $m \times n$, then

(a) $A\mathbf{x} = \mathbf{b}$ will be solvable for every $\mathbf{b}$ if the (circle one) *column space* / *nullspace* of A is $\underline{\mathbb{R}^m}$.
This happens when the rank of A is $\underline{m}$.

(b) $A\mathbf{x} = \mathbf{b}$ will have at most one solution if the (circle one) *column space* / *nullspace* of A is $\underline{\{\vec{0}\}}$.
This happens when the rank of A is $\underline{n}$.

3. (13 pts.) For

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

either find A^{-1}, or show it does not exist.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ -2 & -2 & -1 \end{pmatrix}$$

4. (10 pts.) Determine whether the following vectors in $\mathbb{R}^4$ are independent.

$$(1, 0, -1, 1), (-1, 1, 2, 1), (1, 1, 2, 3)$$

Solution 1: We solve $c_1\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \vec{0}$

i.e. $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}\vec{c} = \vec{0}$ $\quad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$

Since there is a pivot in every column, the only solution is $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so the vectors are independent.

Solution 2: Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ so the row space of A is the span of the 3 vectors

$A \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$. Since there are 3 pivots, the rowspace is 3-dimensional, so the vectors must be independent

5. (18 pts. – 4 pts. each M, 2 pts. each M^{-1}) Give matrices M that perform the following operation to a 3×3 matrix A when MA is computed. Also, give M^{-1}.

(a) Reorder the rows of A so that the 3rd is on top, the 1st in the middle, and the 2nd at the bottom:

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad M^{-1} = M^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) Multiply rows 1, 2, and 3 by the scalars 2, 4, and 8, respectively:

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{pmatrix}$$

(c) Add twice the first row to the third:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

6. (10 pts. – 2 pts. each) Suppose A is 4×3, and when $\mathbf{b} = (-1, 1, 2, -2)$ the solutions to $A\mathbf{x} = \mathbf{b}$ form a line. Then

(a) The solutions (circle one) *form* / *do not form* a subspace of $\mathbb{R}^3$.

(b) The rank of A must be 2.

(c) For a randomly chosen $\mathbf{b}$ in $\mathbb{R}^4$ the problem $A\mathbf{x} = \mathbf{b}$ (circle one) *will have* / *probably will have* / *probably will not have* / *will not have* a solution.

(d) The nullspace of A is a (circle one) *point* / *line* / *plane* / *3-space*/ *4-space*.

(e) The row space of A is a (circle one) *point* / *line* / *plane* / *3-space*/ *4-space*.

7. (10 pts.) Find the LDL^T factorization of $A = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 3 \\ 0 & -7/2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -7/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 3/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -7/2 \end{pmatrix} \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}$$

8. (8 pts. – 4 pts. each) Short answers:

(a) Give the formula for the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(b) If the LU factorization of an $n \times n$ matrix A is known, then $A\mathbf{x} = \mathbf{b}$ can be solved by solving what triangular systems? $L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$. First solve $L\vec{y} = \vec{b}$ for $\vec{y}$, and then solve $U\vec{x} = \vec{y}$ for $\vec{x}$.