Math 314 Name :__________________

Final Exam December 17, 2014

Show your work on all problems.

1. (10 pts.) For

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

(a) (6 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

(b) (2 pts.) For a different vector $\mathbf{b}$, could $A\mathbf{x} = \mathbf{b}$ have no solutions? Briefly explain why.

(c) (2 pts.) For a different vector $\mathbf{b}$, could $A\mathbf{x} = \mathbf{b}$ have exactly one solution? Briefly explain why.

2. (6 pts.) Using elimination, find A^{-1}, or show it doesn't exist, for $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

3. (7 pts.) Elimination on A produces the reduced row echelon matrix U, where

$$A = \begin{pmatrix} 1 & 3 & 1 & 8 \\ 1 & 3 & 0 & 2 \\ -1 & -3 & 1 & 4 \\ 2 & 6 & 1 & 11 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (1 pt.) What is the rank of A?

(b) (2 pts.) What is a basis for the row space of A?

(c) (2 pts.) What is a basis for the column space of A?

(d) (2 pts.) What is a basis for the nullspace space of A?

4. (7 pts.) Let V be the vector space of polynomials of degree at most 3, with basis $1, x, x^2, x^3$. Consider the transformation $T : V \to \mathbb{R}^1$ that evaluates a polynomial in V at $x = 2$.
(For instance, $T(1 - x + 3x^3) = 1 - 2 + 3(2)^3 = 23$.)

(a) (3 pts.) Explain why T is a linear transformation.

(b) (4 pts.) Using the above basis for V and the basis $e_1 = 1$ for $\mathbb{R}^1$, give the 1×4 matrix expressing T.

5. (10 pts.) Find a diagonalization of $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ that uses an *orthogonal* matrix.

6. (6 pts.) Suppose A is a 3×5 matrix.

   (a) (2 pts.) If A is a randomly chosen matrix, what are the most likely dimensions of the nullspace and columnspace?

   (b) (4 pts.) If for some particular vector $\mathbf{b} \in \mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ has no solutions, what are the possible dimensions of the nullspace and column space?

7. (7 pts.) Find the volume of the parallelepiped (or "box") with 3 edges given by $(1, -1, 0)$, $(1, 1, 1)$, and $(0, 1, -1)$.

8. (12 pts.—2 pts. each) A matrix A has singular value decomposition $A = U\Sigma V^T$ with

$$U = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a \\ 0 & 1/\sqrt{2} & b \end{pmatrix}$$

and singular values $\sigma_1 = 5, \sigma_2 = 3, \sigma_3 = 2$.

(a) Give Σ. (Be careful that your answer has the right size.)

(b) What are all possibilities for the numbers a, b in column 3 of V? (If you knew A, there would be only one possibility, but A is not given here).

(c) Give a basis for the the column space of A.

(d) Give a basis for the row space of A

(e) The columns of U are eigenvectors of what matrix?

(f) Give the pseudoinverse A^+, as a product of matrices.

9. (6 pts.) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (0, 1, 0, 1)$, $\mathbf{v}_2 = (1, -1, 1, 1)$, and $\mathbf{v}_3 = (1, 1, 1, 1)$.

10. (8 pts.—2 pts. each) Give matrices that perform the following operations on vectors:

    (a) Reorder the entries so that (a, b, c) becomes (c, b, a).

    (b) Rotate a vector in $\mathbb{R}^2$ counterclockwise by angle θ.

    (c) Subtract 5 times the top entry from the third entry of a vector in $\mathbb{R}^4$.

    (d) Project vectors in $\mathbb{R}^2$ onto the line spanned by $(1, -1)$.

11. (20 pts.) Complete the following.

(a) (2 pts.) If A is $m \times n$, then $A\mathbf{x} = \mathbf{b}$ will be solvable for every $\mathbf{b}$ if the rank of A is ______________.

(b) (4 pts.) If $A\mathbf{x} = \mathbf{b}$ has no solution, the least-squares best-fit solution can be found by solving ________________. This is equivalent to replacing $\mathbf{b}$ with ______________ in the original equation.

(c) (2 pts.) The definition of the *dimension* of a space is...

(d) (2 pts.) If P is a matrix that projects vectors onto a subspace V of $\mathbb{R}^n$, then $I - P$ is a matrix that...

(e) (4 pts.) The value of $7x^2 + 4xy + y^2$ IS / IS NOT (choose one) always positive for $(x, y) \neq (0, 0)$ because it can be expressed by the symmetric matrix ______________ and ...

(f) (2 pts.) The definition of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ being *linearly independent* is...

(g) (4 pts.) Before attempting to find the inverse of a large square matrix, it would be nice to know its determinant, since the inverse exists exactly when the determinant of A is ______________. However, it is generally not worthwhile to calculate the determinant in this situation since...