SOLUTIONS

Math 314 Name : ________________

Final Exam December 17, 2014

Show your work on all problems.

1. (10 pts.) For

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

(a) (6 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

$$\rightarrow \left(\begin{array}{cccc|c} 2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = s,\ x_4 = t$ (free)

$2x_1 + x_2 + x_3 = -1$

$x_3 + x_4 = 1$

$x_3 = 1 - t$

$2x_1 + s + (1-t) = -1$

$x_1 = \frac{1}{2}(-1-1+t-s)$

$= -1 + \frac{1}{2}t - \frac{1}{2}s$

$$\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

(b) (2 pts.) For a different vector $\mathbf{b}$, could $A\mathbf{x} = \mathbf{b}$ have no solutions? Briefly explain why.

Yes. Because row of zeros we know $N(A^T) \neq Z$, so if $\vec{b}$ in $N(A^T)$ then no solns

(c) (2 pts.) For a different vector $\mathbf{b}$, could $A\mathbf{x} = \mathbf{b}$ have exactly one solution? Briefly explain why.

No. $N(A) \neq Z$ so there are multiple solutions if there is one solution.

2. (6 pts.) Using elimination, find A^{-1}, or show it doesn't exist, for $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Gauss-Jordan

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -2 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

3. (7 pts.) Elimination on A produces the reduced row echelon matrix U, where

$$A = \begin{pmatrix} 1 & 3 & 1 & 8 \\ 1 & 3 & 0 & 2 \\ -1 & -3 & 1 & 4 \\ 2 & 6 & 1 & 11 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (1 pt.) What is the rank of A?

$\text{rank} = 3$

nonzero rows of U

(b) (2 pts.) What is a basis for the row space of A?

$$C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) (2 pts.) What is a basis for the column space of A?

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \\ 4 \\ 11 \end{bmatrix} \right\}$$

pivot columns of A

(d) (2 pts.) What is a basis for the nullspace space of A?

x_2 free $\Rightarrow$

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

4. (7 pts.) Let V be the vector space of polynomials of degree at most 3, with basis 1, x, x^2, x^3. Consider the transformation $T : V \to \mathbb{R}^1$ that evaluates a polynomial in V at $x = 2$. (For instance, $T(1 - x + 3x^3) = 1 - 2 + 3(2)^3 = 23$.)

(a) (3 pts.) Explain why T is a linear transformation.

if $p(x)$, $q(x)$ are polynomials then $T(ap(x) + bq(x)) = ap(2) + bq(2) = aT(p) + bT(q)$; evaluation is a linear process

(b) (4 pts.) Using the above basis for V and the basis $e_1 = 1$ for $\mathbb{R}^1$, give the 1×4 matrix expressing T.

$$T\left(\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}\right) = \alpha + 2\beta + 4\gamma + 8\delta$$

because $p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$

$p(2) = \alpha + \beta \cdot 2 + \gamma \cdot 4 + \delta \cdot 8$

so $T\left(\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$

5. (10 pts.) Find a diagonalization of $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ that uses an *orthogonal* matrix.

symmetric

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(1-\lambda) - 9 = \lambda^2 - 2\lambda - 8$$

$$= (\lambda - 4)(\lambda + 2) \quad \therefore \quad \lambda_1 = -2, \ \lambda_2 = 4$$

$\lambda_1 = -2$: $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \quad \therefore \quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \therefore \quad \vec{q}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\lambda_2 = 4$: $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \quad \therefore \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \quad \vec{q}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore \quad A = Q \Lambda Q^T$ where $Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$

6. (6 pts.) Suppose A is a 3×5 matrix.

(a) (2 pts.) If A is a randomly chosen matrix, what are the most likely dimensions of the nullspace and columnspace?

typically:
$\dim C(A) = 3$, $\dim N(A) = 2$ (rank=2)

(b) (4 pts.) If for some particular vector $\mathbf{b} \in \mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ has no solutions, what are the possible dimensions of the nullspace and column space?

$\dim C(A) = 0, 1, 2$

$\dim N(A) = 5, 4, 3$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $\dim C(A) + \dim N(A) = \dim C(A^T) + \dim N(A) = 5$

7. (7 pts.) Find the volume of the parallelepiped (or "box") with 3 edges given by $(1, -1, 0)$, $(1, 1, 1)$, and $(0, 1, -1)$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow (\text{volume}) = |\det(A)|$$

$$= |1(-1-1) - 1(1)| = |-2-1| = 3$$

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8. (12 pts.—2 pts. each) A matrix A has singular value decomposition $A = U\Sigma V^T$ with

$$U = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a \\ 0 & 1/\sqrt{2} & b \end{pmatrix} \Big\} \text{ A is } 4\times 3$$

and singular values $\sigma_1 = 5, \sigma_2 = 3, \sigma_3 = 2$.

(a) Give Σ. (Be careful that your answer has the right size.)

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) What are all possibilities for the numbers a, b in column 3 of V? (If you knew A, there would be only one possibility, but A is not given here).

since $\vec{v}_3 \perp \vec{v}_2$: $\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b = 0 \quad \therefore \quad a+b=0$ <u>and</u> $a^2+b^2=1$

so either $a = \frac{-1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}$ or $a = \frac{1}{\sqrt{2}}, b = \frac{-1}{\sqrt{2}}$

(c) Give a basis for the the column space of A.

$\vec{u}_1, \vec{u}_2, \vec{u}_3$

(d) Give a basis for the row space of A

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

(e) The columns of U are eigenvectors of what matrix?

AA^T

$$\left(\begin{aligned} AA^T &= U\Sigma V^T(U\Sigma V^T)^T \\ &= U\Sigma V^T V \Sigma^T U^T \\ &= U\underbrace{(\Sigma\Sigma^T)}_{\Lambda}U^T \end{aligned} \right)$$

(f) Give the pseudoinverse A^+, as a product of matrices.

A^+ is 3×4; $A^+A = I$

$$A^+ = V \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} U^T$$

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9. (6 pts.) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (0, 1, 0, 1)$, $\mathbf{v}_2 = (1, -1, 1, 1)$, and $\mathbf{v}_3 = (1, 1, 1, 1)$.

$$\vec{q}_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{w}_2 = \vec{v}_2 - (\vec{q}_1^T \vec{v}_2)\vec{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - 0\vec{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \vec{q}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{q}_1^T \vec{v}_3)\vec{q}_1 - (\vec{q}_2^T \vec{v}_3)\vec{q}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{2}}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - 1\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\vec{q}_3 = \vec{w}_3$$

$$\therefore S = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\left\{ \underset{\vec{q}_1}{\underset{\parallel}{\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}}, \underset{\vec{q}_2}{\underset{\parallel}{\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}}, \underset{\vec{q}_3}{\underset{\parallel}{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}}} \right\}$$

10. (8 pts.—2 pts. each) Give matrices that perform the following operations on vectors:

(a) Reorder the entries so that (a, b, c) becomes (c, b, a).

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Rotate a vector in $\mathbb{R}^2$ counterclockwise by angle θ.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\left(\underline{\theta = \pi/2}: \ R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right.$ ✓

$\left. R\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, R\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

(c) Subtract 5 times the top entry from the third entry of a vector in $\mathbb{R}^4$. ⇐ but otherwise leave it unchanged

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) Project vectors in $\mathbb{R}^2$ onto the line spanned by $(1, -1)$.

$$\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

11. (20 pts.) Complete the following.

(a) (2 pts.) If A is $m \times n$, then $A\mathbf{x} = \mathbf{b}$ will be solvable for every $\mathbf{b}$ if the rank of A is $\underline{\quad m \quad}$.

(b) (4 pts.) If $A\mathbf{x} = \mathbf{b}$ has no solution, the least-squares best-fit solution can be found by solving $\underline{A^T A\vec{x} = A^T\vec{b}}$. This is equivalent to replacing $\mathbf{b}$ with $\underline{\quad P\vec{b} \quad}$ in the original equation.

(c) (2 pts.) The definition of the *dimension* of a space is...

the number of vectors in a basis for the space

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(d) (2 pts.) If P is a matrix that projects vectors onto a subspace V of $\mathbb{R}^n$, then $I - P$ is a matrix that...

projects onto the orthogonal complement of V

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(e) (4 pts.) The value of $7x^2 + 4xy + y^2$ IS / IS NOT (choose one) always positive for $(x, y) \neq (0, 0)$ because it can be expressed by the symmetric matrix $\underline{A = \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}}$ and A is positive def.

$$7x^2 + 4xy + y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(f) (2 pts.) The definition of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ being *linearly independent* is...

if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$ then $c_1 = 0, c_2 = 0, \ldots, c_k = 0$

(g) (4 pts.) Before attempting to find the inverse of a large square matrix, it would be nice to know its determinant, since the inverse exists exactly when the determinant of A is $\underline{\text{nonzero}}$. However, it is generally not worthwhile to calculate the determinant in this situation since...

the determinant is likely to be very large or very small, and overflow or underflow is common in floating point arithmetic