Math 314 Name :__________________

Final Exam May 8, 2013

Show all your work.

1. (10 pts.) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

   (a) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

   Use your answer to (a) to answer the following, without additional calculation:

   (b) Give a basis for the nullspace of A.

   (c) Give a basis for the columnspace of A.

2. (6 pts.)

   (a) (2 pts.) Give a symmetric matrix A so that $\mathbf{x}^T A\mathbf{x} = 2x^2 - 2xy + y^2$

   (b) (4 pts.) Determine whether A is positive definite. (This can be done several different ways; any correct method is acceptable as long as your work is shown.)

3. (8 pts.) The vectors $\mathbf{v}_1 = (1,0,0,0), \mathbf{v}_2 = (-2,1,0,2), \mathbf{v}_3 = (-1,3,0,1)$ are a basis for a subspace of $\mathbb{R}^4$. Find an orthonormal basis for that subspace.

4. (10 pts.) Find inverses of the following matrices, if they exist. If no inverse exists, explain how you know that.

   (a) (3 pts.) $\begin{pmatrix} 1 & 4 \\ -1 & -2 \end{pmatrix}$

   (b) (7 pts.) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -3 \\ 1 & 2 & 1 \end{pmatrix}$

5. (10 pts.) The annual changes in the population of an organism is modeled by $\mathbf{p}_{t+1} = A\mathbf{p}_t$, where $A = \begin{pmatrix} 0 & 3 \\ 0.2 & 0.7 \end{pmatrix}$. The first entry of $\mathbf{p}_t$ refers to young, and the second to adults.

(a) (2 pts.) What is the biological meaning of the 0.2 in this matrix?

(b) (2 pts.) The diagonalization $A = S\Lambda S^{-1}$ has $\Lambda = \begin{pmatrix} 1.2 & 0 \\ 0 & -.5 \end{pmatrix}$ and $S = \begin{pmatrix} 5 & 6 \\ 2 & -1 \end{pmatrix}$. What is the stable age distribution of this model? What does it tell you about the population?

(c) (3 pts.) Sketch a graph of time vs. population size for the two groups in this model that indicates typical qualitative behavior you should see if the initial population, $\mathbf{p}_0$, is randomly chosen.

(d) (3 pts.) If the initial population were $\mathbf{p}_0 = (10, 4)$ (twice the first column of S), what would $\mathbf{p}_{20}$ be?

6. (8 pts. — 4 pts. each) Decide whether each of these transformations $T : \mathbb{R}^2 \to R^2$ is linear or not. If it is not linear, explain why it is not. If it is linear, give a matrix to express it (using the standard bases).

(a) $T(x, y) = (x, y) + (1, -1)$.

(b) $T(x, y) = (y, 0)$.

7. (8 pts. — 4 pts. each) Let $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}$.

(a) Find all eigenvalues of A.

(b) Find an eigenvector for the *largest* eigenvalue of A.

8. (8 pts.) Let $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

(a) (6 pts.) Let V be the column space of A. Find a basis for $V^{\perp}$.

(b) (2 pts.) What is the rank of A? (If you did part (a) you should be able to answer this with no further computation.)

9. (8 pts.) Four data points in the plane are $(x, y) = (-2, 2)$, $(-1, 2)$, $(0, 1)$, $(1, -1)$.

   (a) (2 pts.) Give a matrix equation that you would *like to solve* (but which has no solution) to find the equation of a line $y = mx + b$ through these points.

   (b) (6 pts.) Give a matrix equation that *can be solved* to find the least squares best-fit line for these points. (Do not solve it.)

10. (24 pts. — 2 pts. each) Give short answers.

   (a) We can be sure a real square matrix has real eigenvalues if it is ____________________________

   (b) If for some specific $A, \mathbf{b}$, where A is 4×5 and $\mathbf{b} \in \mathbb{R}^4$, we know $A\mathbf{x} = \mathbf{b}$ has no solution, then the rank of A must be (list all possibilities) ____________________________

   (c) If for some specific $A, \mathbf{b}$, where A is 4×5 and $\mathbf{b} \in \mathbb{R}^4$, we know $A\mathbf{x} = \mathbf{b}$ has a 2-dimensional plane of solutions, then the rank of A must be (list all possibilities) ____________________________

   (d) If for some specific A, where A is 4×5 we know $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, then the rank of A must be (list all possibilities) ____________________________

   (e) If Q is an orthogonal matrix, then its determinant can only be __________.
   (Hint: What is $Q^T Q$?)

   (f) Using the 'big formula' to find the determinant of a 7×7 matrix would requiring adding (or subtracting) __________ terms, each of which is the product of __________ numbers. For such a matrix, it would be much easier to compute the determinant by ____________________________

(g) Using Cramer's rule to solve $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$ gives $x =$

(h) A 4×4 matrix with eigenvalues 1,3,3,3 could have as few as ___________ eigenvectors. But it must have 4 orthogonal eigenvectors if A is ______________________________

(i) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ form three edges of a parallelopiped, then its volume can be computed as _______________

(j) A matrix that rotates vectors in $\mathbb{R}^2$ about the origin by an angle of θ counterclockwise is $R_\theta =$

(k) If the SVD of a matrix A is

$$U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

then the rank of A is ___________ and the pseudoinverse A^+ is (you may leave your answer as a product):

(l) If A has eignevector $\mathbf{v}$ with eigenvalue λ, then MAM^{-1} will have eigenvector ___________ with eigenvalue ___________.