Homework #11

Due Friday 15 April, 2022 at 11:59pm. ← *REVISED*

Submit as a single PDF via Gradescope; see the Canvas page
canvas.alaska.edu/courses/7017
Textbook Problems from Strang, *Intro Linear Algebra*, 5th ed. will be graded for completion. Answers/solutions to these Problems are linked at
bueler.github.io/math314/resources.html
The **P** Problems will be graded for correctness. When grading these Problems, I will expect you to write explanations using complete sentences!

Put these Textbook Problems first on your PDF, in this order.

from Problem Set 6.1, pages 297–302: # 1, 5, 8, 14, 16, 21, 24, 33

from Problem Set 6.2, pages 313–317: # 1, 2, 4, 7, 15, 20

Put these ***P*** *Problems next on your PDF, in this order.*

P52. **(a)** Compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of the matrices the matrices $A + I$ and A^{-1}. Confirm that the eigenvectors are the same as those for A.

(c) Apply the functions $f(x) = x + 1$ and $g(x) = 1/x$ to the eigenvalues of A, and confirm this gives the same eigenvalues computed in **(b)**.

This illustrates a more general rule, that functions of a matrix, like $f(A) = A + I$ and $g(A) = A^{-1}$, have eigenvalues which are computed by the same function, but on complex numbers.

P53. **(a)** Compute the eigenvalues of the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

(b) The matrix B is rank one. To consider this case, suppose $\boldsymbol{u}, \boldsymbol{v}$ are nonzero (column) vectors in $\mathbb{R}^n$. Let $C = \boldsymbol{u}\boldsymbol{v}^\top$. (Recall this is the general formula for a rank one matrix!) Show that $C\boldsymbol{u} = \lambda \boldsymbol{u}$ and find λ. Also show that if $\boldsymbol{w}$ is any vector orthogonal to $\boldsymbol{v}$ then $C\boldsymbol{w} = \boldsymbol{0}$. What can you conclude about the eigenvalues of rank one matrices?

P54. 2 by 2 rotation matrices have the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Using this form, find a non-identity matrix A with the property that $A^3 = I$. Now compute the eigenvalues of A, and confirm that the eigenvalues are certain complex numbers λ, on the unit circle, which satisfy $\lambda^3 = 1$.

P55. Answer true or false; if true give an explanation and if false give a counterexample. Assume A is 3 by 3.

(a) If the eigenvalues of A are $2, 2, 5$ then A is invertible.

(b) If the eigenvalues of A are $2, 2, 5$ then A is diagonalizable.

P56. The following four diagonalizable 2×2 matrices look rather similar to each other, for instance in terms of the sizes of the entries:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

However, their *powers* act in completely different ways. Looking at the eigenvalues will explain it.

(a) Compute the eigenvalues of the 4 matrices and put all 8 values on a single graph in the complex plane. (*The eigenvalues of real matrices are generally complex!*) Show the unit circle on your graph, and then show the eigenvalues as 8 clearly-labeled dots in the complex plane; consider using a color for each matrix. Draw the picture well enough so that you can tell where all the eigenvalues are relative to the unit circle.

(b) Recall that if M is a square and diagonalizable matrix with eigenvalues λ_i and eigenvectors $\boldsymbol{x}_i$, so that $M = X\Lambda X^{-1}$ is the diagonalization, then

$$M^k = X\Lambda^k X^{-1}.$$

(*Remember the calculation:* $M^2 = X\Lambda X^{-1} X\Lambda X^{-1} = X\Lambda^2 X^{-1}$.) That is, powers of A can be computed by putting the power on the diagonal matrix Λ.

Based on where the eigenvalues in part **(a)** are, relative to the unit circle, match A, B, C, D to these descriptions:

1. $M^{100} = I$
2. $M^{100} = -M$
3. M^{100} has tiny entries
4. M^{100} has enormous entries

(*Hint.* A power of a complex number $x + iy = re^{i\theta}$ which is on the unit circle just rotates around the circle. In general, $(re^{i\theta})^k = r^k e^{ik\theta}$.)

(c) Use Matlab to compute the 100th power of the 4 matrices to confirm your results.